Semi-classical Orthogonal Polynomials and the Painlevé Equations

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Athena SWAN Bronze Award

Alternative discrete Painlevé I equation

$$x_n + x_{n+1} = y_n^2 - t \qquad x_0(t) = 0, \quad y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$$
$$x_n(y_n + y_{n-1}) = n$$

Second Painlevé equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + A$$

with A a constant.

References

- **P A Clarkson, A F Loureiro & W Van Assche**, "Unique positive solution for the alternative discrete Painlevé I equation", *Journal of Difference Equations and Applications*, DOI: 10.1080/10652469.2015.1098635 (2016)
- **P A Clarkson**, "On Airy Solutions of the Second Painlevé Equation", *Studies in Applied Mathematics*, DOI: 10.1111/sapm.12123 (2016)

Painlevé Equations

$$\frac{d^2q}{dz^2} = 6q^2 + z \qquad \mathrm{P_I}$$

$$\frac{d^2q}{dz^2} = 2q^3 + zq + A \qquad \mathrm{P_{II}}$$

$$\frac{d^2q}{dz^2} = \frac{1}{q}\left(\frac{dq}{dz}\right)^2 - \frac{1}{z}\frac{dq}{dz} + \frac{Aq^2 + B}{z} + Cq^3 + \frac{D}{q} \qquad \mathrm{P_{III}}$$

$$\frac{d^2q}{dz^2} = \frac{1}{2q}\left(\frac{dq}{dz}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q} \qquad \mathrm{P_{IV}}$$

$$\frac{d^2q}{dz^2} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{dq}{dz}\right)^2 - \frac{1}{z}\frac{dq}{dz} + \frac{(q-1)^2}{z^2}\left(Aq + \frac{B}{q}\right) + \frac{Cq}{z} + \frac{Dq(q+1)}{q-1} \qquad \mathrm{P_V}$$

$$\frac{d^2q}{dz^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-z}\right)\left(\frac{dq}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{q-z}\right)\frac{dq}{dz} + \frac{q(q-1)(q-z)}{z^2(z-1)^2}\left\{A + \frac{Bz}{q^2} + \frac{C(z-1)}{(q-1)^2} + \frac{Dz(z-1)}{(q-z)^2}\right\} \qquad \mathrm{P_{VI}}$$

with A, B, C and D arbitrary constants.

Special function solutions of Painlevé equations

	Number of (essential) parameters	Special function	Number of parameters	Associated orthogonal polynomial
P_{I}	0	—		
P_{II}	1	**Airy** $\mathrm{Ai}(z), \mathrm{Bi}(z)$	0	—
P_{III}	2	**Bessel** $J_\nu(z), I_\nu(z), K_\nu(z)$	1	—
P_{IV}	2	**Parabolic** $D_\nu(z)$	1	**Hermite** $H_n(z)$
P_{V}	3	**Kummer** $M(a,b,z), U(a,b,z)$ **Whittaker** $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$	2	**Associated Laguerre** $L_n^{(k)}(z)$
P_{VI}	4	**hypergeometric** ${}_2F_1(a,b;c;z)$	3	**Jacobi** $P_n^{(\alpha,\beta)}(z)$

Monic Orthogonal Polynomials

Let $P_n(x)$, $n = 0, 1, 2, \ldots$, be the **monic orthogonal polynomials** of degree n in x, with respect to the positive weight $\omega(x)$, such that

$$\int_a^b P_m(x)P_n(x)\,\omega(x)\,\mathrm{d}x = h_n\delta_{m,n}, \quad h_n > 0, \qquad m, n = 0, 1, 2, \ldots$$

One of the important properties that orthogonal polynomials have is that they satisfy the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

where the recurrence coefficients are given by

$$\alpha_n = \frac{\widetilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\widetilde{\Delta}_n}{\Delta_n}, \qquad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}$$

with

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-1} \\ \mu_1 & \mu_2 & \ldots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-2} \end{vmatrix}, \qquad \widetilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \ldots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}$$

and $\mu_k = \int_a^b x^k\,\omega(x)\,\mathrm{d}x$ are the **moments** of the weight $\omega(x)$.

Semi-classical Orthogonal Polynomials

Consider the **Pearson equation** satisfied by the weight $\omega(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sigma(x)\omega(x)] = \tau(x)\omega(x)$$

- **Classical orthogonal polynomials**: $\sigma(x)$ and $\tau(x)$ are polynomials with $\deg(\sigma) \le 2$ and $\deg(\tau) = 1$

	$\omega(x)$	$\sigma(x)$	$\tau(x)$
Hermite	$\exp(-x^2)$	1	$-2x$
Laguerre	$x^\nu \exp(-x)$	x	$1+\nu-x$
Jacobi	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$	$\beta-\alpha-(2+\alpha+\beta)x$

- **Semi-classical orthogonal polynomials**: $\sigma(x)$ and $\tau(x)$ are polynomials with either $\deg(\sigma) > 2$ or $\deg(\tau) > 1$

	$\omega(x)$	$\sigma(x)$	$\tau(x)$
Airy	$\exp(-\frac{1}{3}x^3 + tx)$	1	$t-x^2$
semi-classical Hermite	$\lvert x\rvert^\nu \exp(-x^2+tx)$	x	$1+\nu+tx-2x^2$
Generalized Freud	$\lvert x\rvert^{2\nu+1} \exp(-x^4+tx^2)$	x	$2\nu+2+2tx^2-4x^4$

If the weight has the form

$$\omega(x;t) = \omega_0(x)\exp(tx)$$

where the integrals $\displaystyle\int_{-\infty}^{\infty} x^k\omega_0(x)\exp(tx)\,\mathrm{d}x$ exist for all $k \geq 0$.

- The recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ satisfy the **Toda system**

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \beta_n - \beta_{n+1}, \qquad \frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n(\alpha_n - \alpha_{n-1})$$

- The kth moment is given by

$$\mu_k(t) = \int_{-\infty}^{\infty} x^k\omega_0(x)\exp(tx)\,\mathrm{d}x = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\int_{-\infty}^{\infty}\omega_0(x)\exp(tx)\,\mathrm{d}x\right) = \frac{\mathrm{d}^k\mu_0}{\mathrm{d}t^k}$$

- Since $\mu_k(t) = \dfrac{\mathrm{d}^k\mu_0}{\mathrm{d}t^k}$, then $\Delta_n(t)$ and $\widetilde{\Delta}_n(t)$ can be expressed as Wronskians

$$\Delta_n(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right) = \det\left[\frac{\mathrm{d}^{j+k}\mu_0}{\mathrm{d}t^{j+k}}\right]_{j,k=0}^{n-1}$$

$$\widetilde{\Delta}_n(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-2}\mu_0}{\mathrm{d}t^{n-2}}, \frac{\mathrm{d}^{n}\mu_0}{\mathrm{d}t^{n}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Delta_n(t)$$

An Alternative Discrete Painlevé I Equation

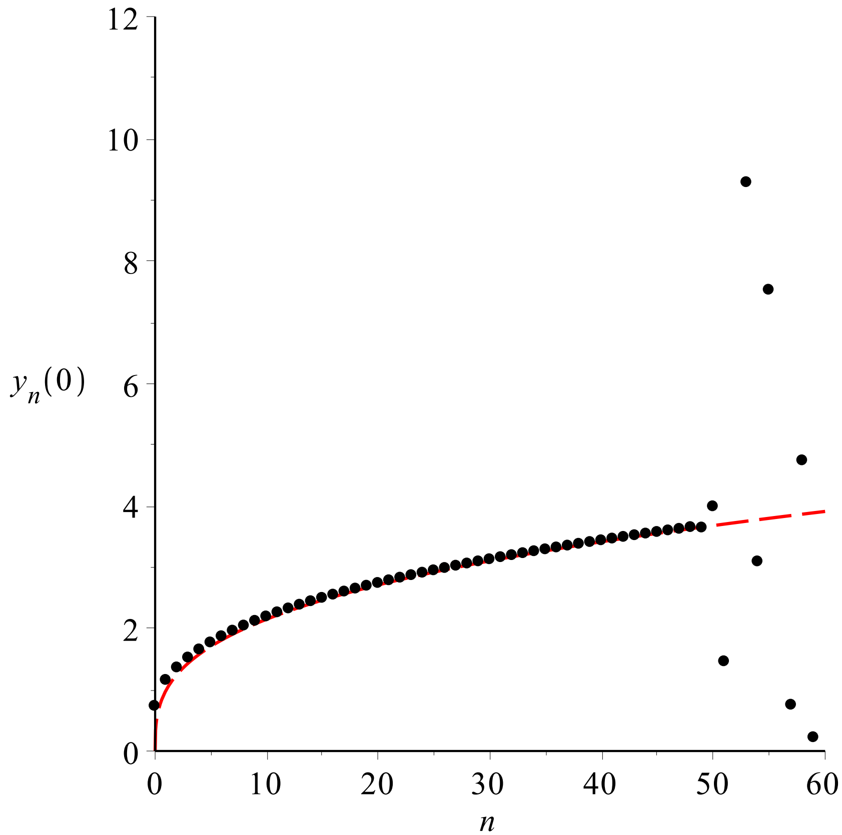
$$x_n + x_{n+1} = y_n^2 - t \qquad x_0(t) = 0, \quad y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$$
$$x_n(y_n + y_{n-1}) = n$$

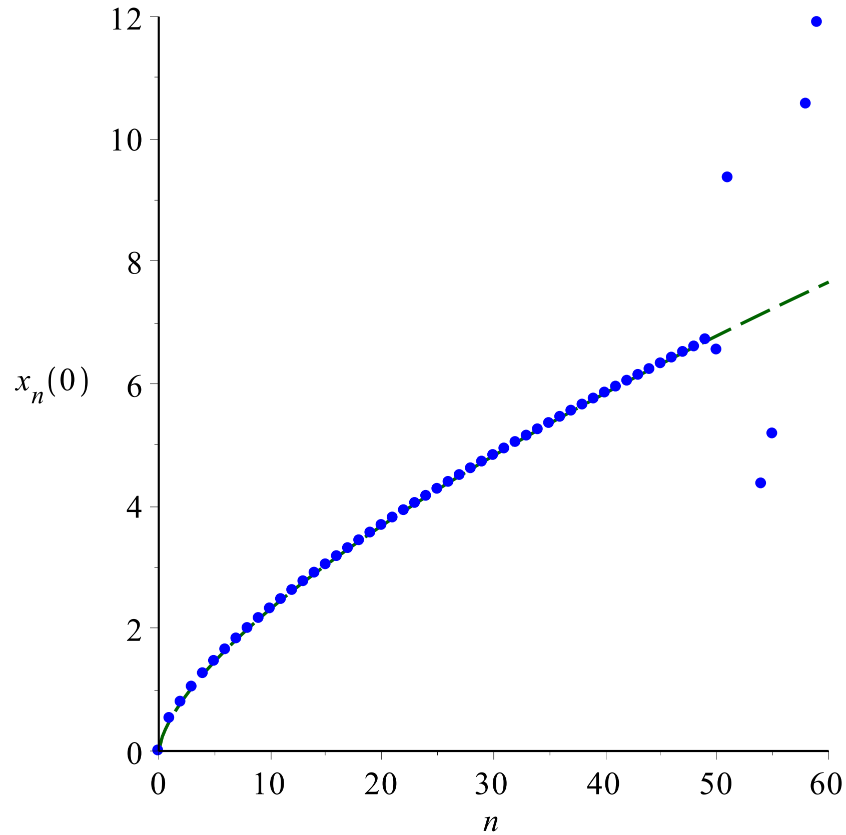
- **PAC, A Loureiro & W Van Assche**, "Unique positive solution for the alternative discrete Painlevé I equation", *Journal of Difference Equations and Applications*, DOI: 10.1080/10652469.2015.1098635 (2016)

$$x_n + x_{n+1} = y_n^2 - t$$
$$x_n(y_n + y_{n-1}) = n$$

$$x_0(t) = 0, \quad y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$$

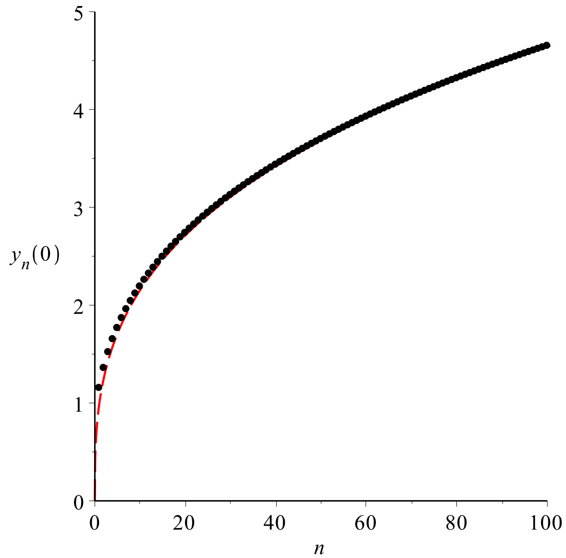
The system is highly sensitive to the initial conditions [50 digits]



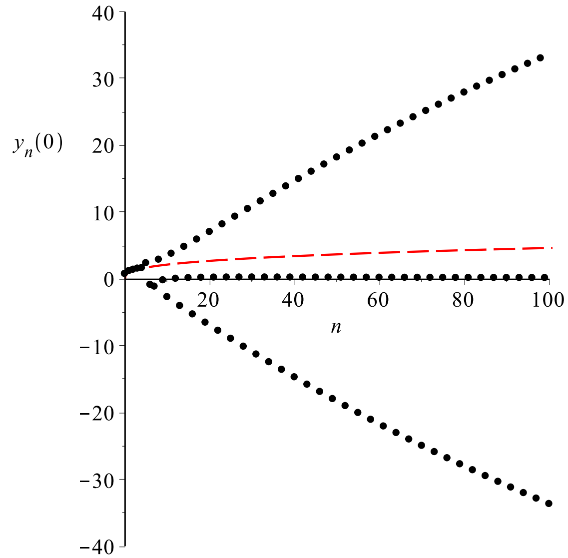


$$y_0(0) = -\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} = 3^{1/3}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}$$

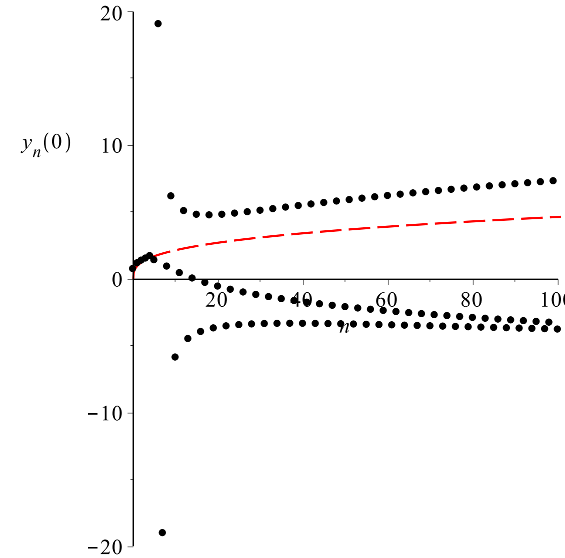
$$x_0(0) = 0$$



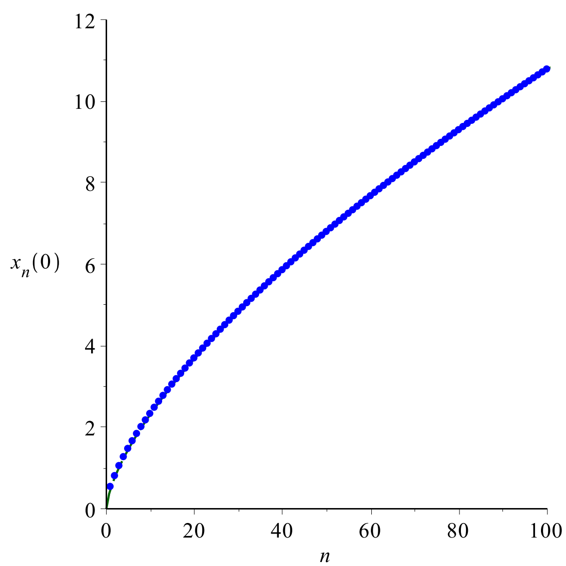
$y_0(0) = 0.7290111...$



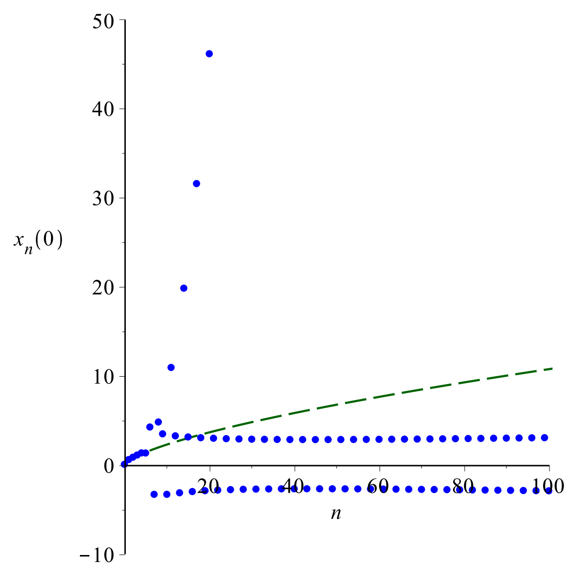
$y_0(0) = 0.729$



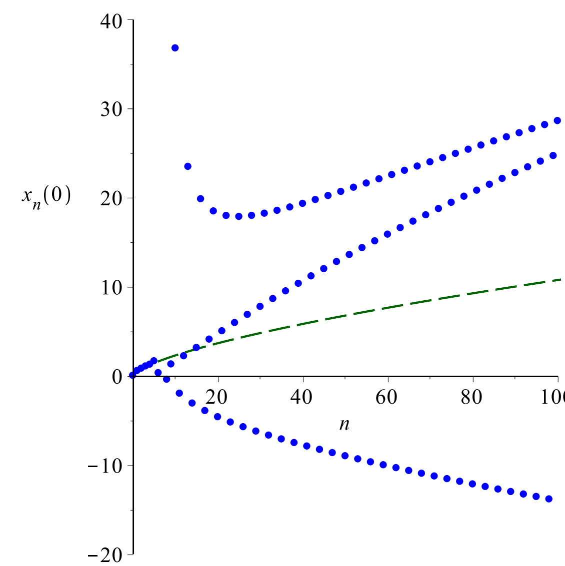
$y_0(0) = 0.72902$



$x_0(0) = 0$



$x_0(0) = 0$



$x_0(0) = 0$

Orthogonal Polynomials on Complex Contours

Consider the semi-classical Airy weight

$$\omega(x;t) = \exp\left(-\tfrac{1}{3}x^3 + tx\right), \qquad t > 0$$

on the curve $\mathcal{C}$ from $\mathrm{e}^{2\pi\mathrm{i}/3}\infty$ to $\mathrm{e}^{-2\pi\mathrm{i}/3}\infty$. The moments are

$$\mu_0(t) = \int_{\mathcal{C}} \exp\left(-\tfrac{1}{3}x^3 + tx\right) \mathrm{d}x = \mathrm{Ai}(t)$$

$$\mu_k(t) = \int_{\mathcal{C}} x^k \exp\left(-\tfrac{1}{3}x^3 + tx\right) \mathrm{d}x = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\,\mathrm{Ai}(t) = \mathrm{Ai}^{(k)}(t)$$

where $\mathrm{Ai}(t)$ is the **Airy function**, the Hankel determinant is

$$\Delta_n(t) = \mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}}\,\mathrm{Ai}(t)\right]_{j,k=0}$$

with $\Delta_0(t) = 1$, and the recursion coefficients are

$$\alpha_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+1}(t)}{\Delta_n(t)}, \qquad \beta_n(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t)$$

with

$$\alpha_0(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\mathrm{Ai}(t) = \frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}, \qquad \beta_0(t) = 0$$

The recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ satisfy the discrete system

$$\begin{aligned}(\alpha_n + \alpha_{n-1})\beta_n - n &= 0\\ \alpha_n^2 + \beta_n + \beta_{n+1} - t &= 0\end{aligned} \tag{1}$$

and the differential system (Toda)

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \beta_{n+1} - \beta_n, \qquad \frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n(\alpha_n - \alpha_{n-1}) \tag{2}$$

Letting $x_n = -\beta_n$ and $y_n = -\alpha_n$ in (1) and (2) yields

$$\begin{aligned}x_n + x_{n+1} &= y_n^2 - t\\ x_n(y_n + y_{n-1}) &= n\end{aligned} \tag{3}$$

which is the discrete system we're interested in, and

$$\frac{\mathrm{d}x_n}{\mathrm{d}t} = x_n(y_{n-1} - y_n), \qquad \frac{\mathrm{d}y_n}{\mathrm{d}t} = x_{n+1} - x_n \tag{4}$$

Then eliminating x_{n+1} and y_{n-1} between (3) and (4) yields

$$\frac{\mathrm{d}y_n}{\mathrm{d}t} = y_n^2 - 2x_n - t, \qquad \frac{\mathrm{d}x_n}{\mathrm{d}t} = -2x_ny_n + n \tag{5}$$

Consider the system

$$\frac{\mathrm{d}y_n}{\mathrm{d}t} = y_n^2 - 2x_n - t, \qquad \frac{\mathrm{d}x_n}{\mathrm{d}t} = -2x_n y_n + n$$

- Eliminating x_n yields

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = 2y_n^3 - 2ty_n - 2n - 1$$

which is equivalent to

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + n + \tfrac{1}{2}$$

i.e. $\mathrm{P_{II}}$ with $A = n + \frac{1}{2}$.

- Eliminating y_n yields

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = \frac{1}{2x_n}\left(\frac{\mathrm{d}x_n}{\mathrm{d}t}\right)^2 + 4x_n^2 + 2tx_n - \frac{n^2}{2x_n}$$

which is equivalent to

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = \frac{1}{2v}\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - 2v^2 - zv - \frac{n^2}{2v}$$

an equation known as $\mathrm{P_{34}}$.

$$\begin{aligned} x_n + x_{n+1} &= y_n^2 - t \\ x_n(y_n + y_{n-1}) &= n \end{aligned} \qquad x_0(t) = 0, \quad y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$$

Solving for x_n yields

$$\frac{n+1}{y_n + y_{n+1}} + \frac{n}{y_n + y_{n-1}} = y_n^2 - t$$

which is known as alt-d$\mathrm{P_I}$ (Fokas, Grammaticos & Ramani [1993]).

We have seen that y_n and x_n satisfy

$$\begin{aligned} \frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} &= 2y_n^3 - 2ty_n - 2n - 1 \\ \frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} &= \frac{1}{2x_n}\left(\frac{\mathrm{d}x_n}{\mathrm{d}t}\right)^2 + 4x_n^2 + 2tx_n - \frac{n^2}{2x_n} \end{aligned}$$

which have "Airy-type" solutions

$$y_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\tau_n(t)}{\tau_{n+1}(t)}, \qquad x_n(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln \tau_n(t)$$

where

$$\tau_n(t) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}} \mathrm{Ai}(t)\right]_{j,k=0}, \quad n \geq 1$$

and $\tau_0(t) = 1$.

Theorem **(PAC, Loureiro & Van Assche [2016])**

For positive values of t, there exists a unique solution of

$$x_n + x_{n+1} = y_n^2 - t$$
$$x_n(y_n + y_{n-1}) = n$$

with $x_0(t) = 0$ for which $x_{n+1}(t) > 0$ and $y_n(t) > 0$ for all $n \geq 0$. This solution corresponds to the initial value

$$y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}.$$

Theorem **(PAC, Loureiro & Van Assche [2016])**

For positive values of t, there exists a unique solution of

$$\frac{n+1}{y_n + y_{n+1}} + \frac{n}{y_n + y_{n-1}} = y_n^2 - t$$

for which $y_n(t) \geq 0$ for all $n \geq 0$. This solution corresponds to the initial values

$$y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}, \qquad y_1(t) = -y_0(t) + \frac{1}{y_0^2(t) - t}$$

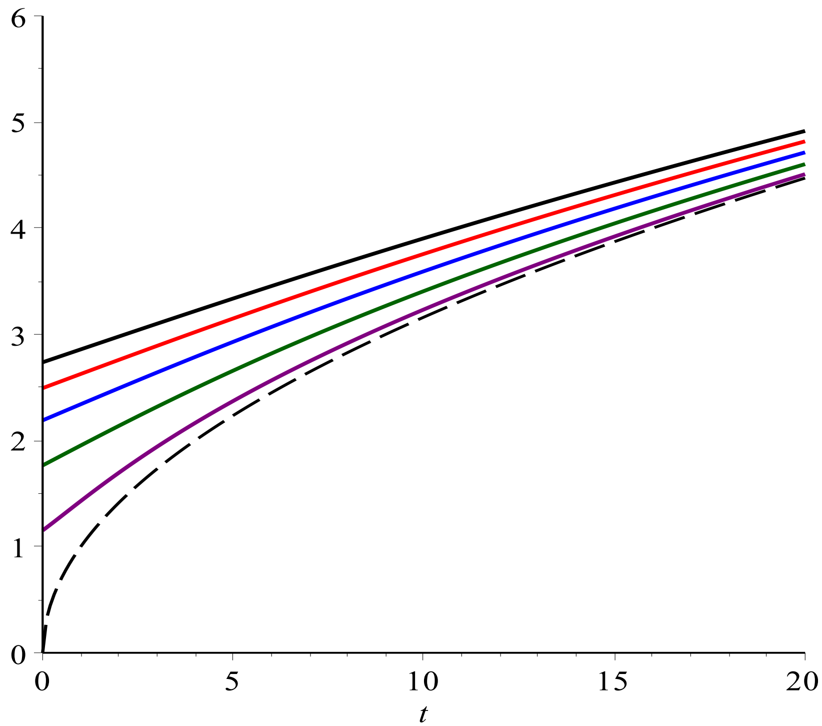
Conjecture *If* $0 < t_1 < t_2$ *then*

$$y_n(t_1) < y_n(t_2), \qquad x_n(t_1) > x_n(t_2)$$

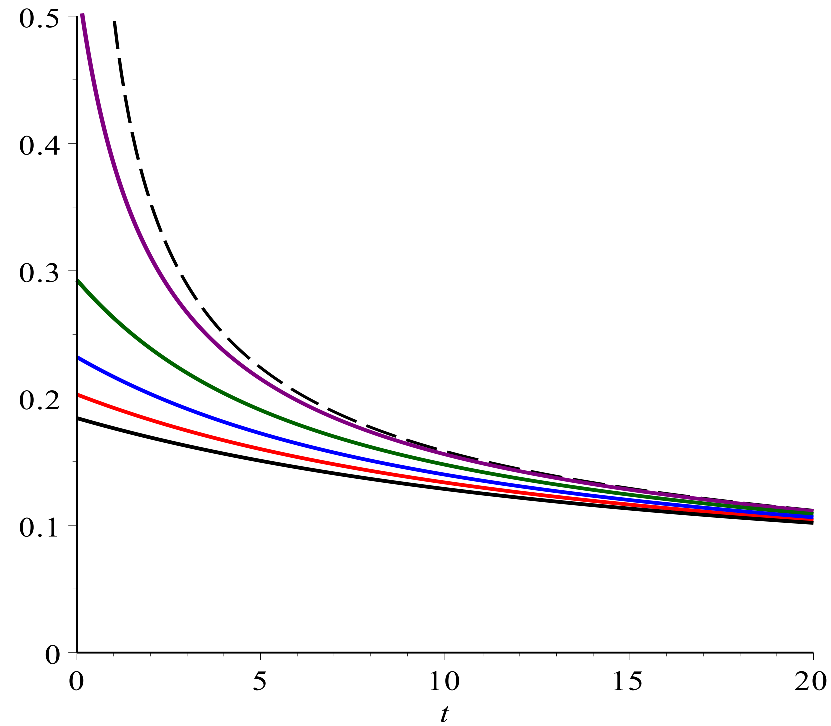
i.e. $y_n(t)$ *is monotonically increasing and* $x_n(t)$ *is monotonically decreasing.*

Conjecture *For fixed* t *with* $t > 0$ *then*

$$\sqrt{t} < y_n(t) < y_{n+1}(t), \qquad \frac{1}{2\sqrt{t}} > \frac{x_n(t)}{n} > \frac{x_{n+1}(t)}{n+1}$$



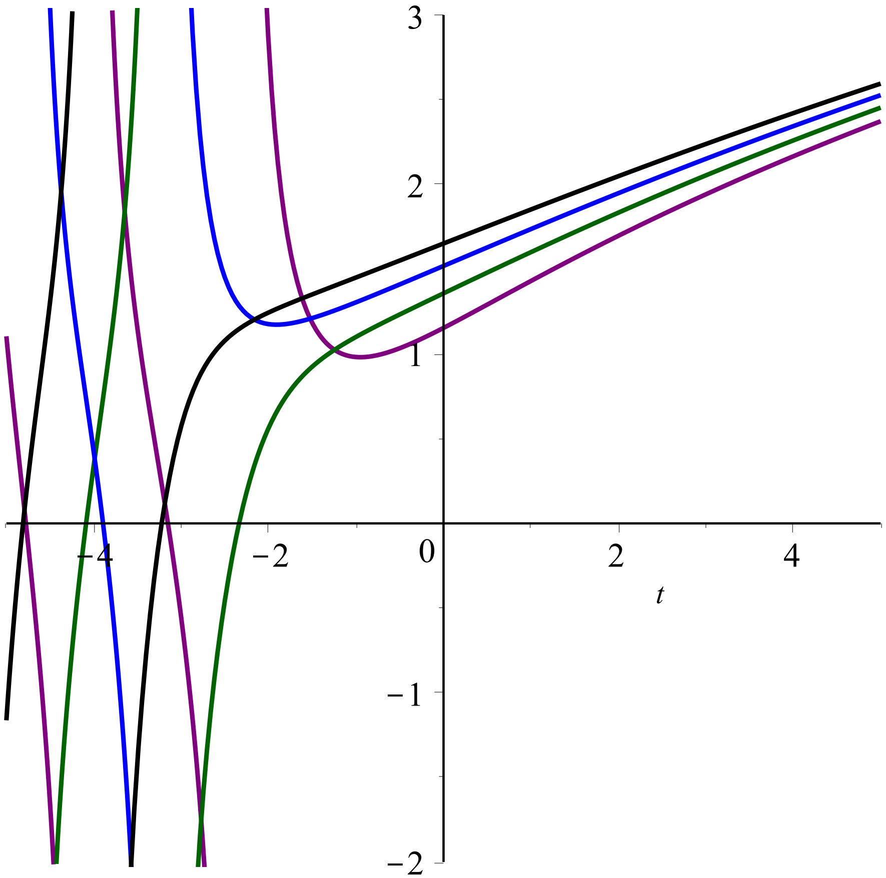
$y_n(t), \quad n = 1, 5, 10, 15, 20$



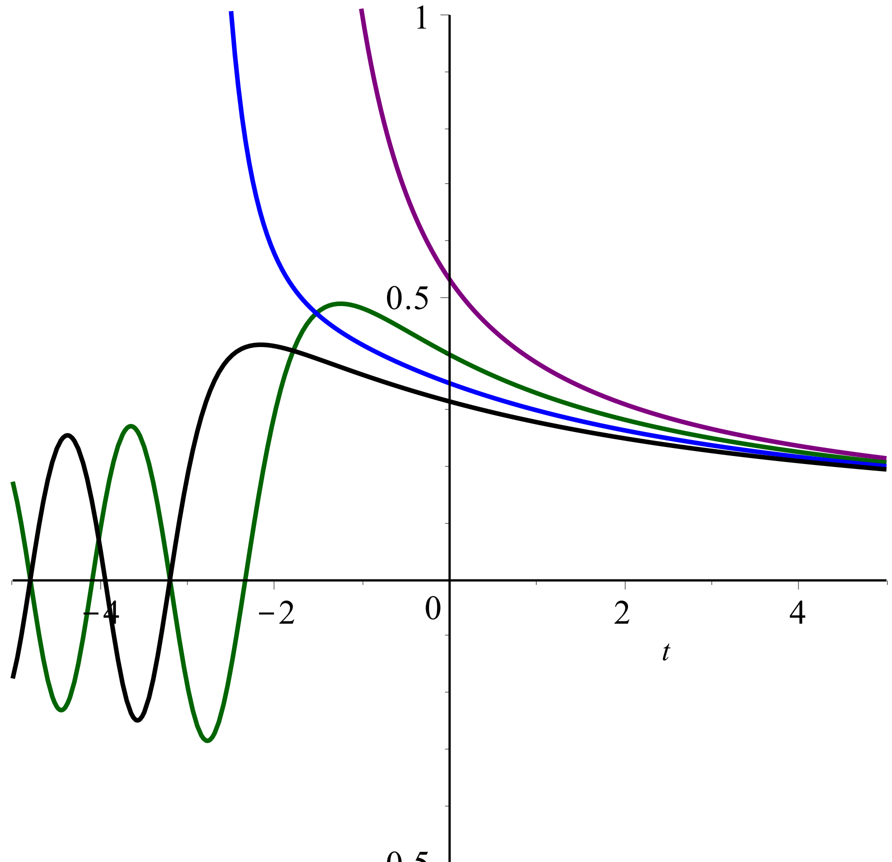
$\frac{1}{n}x_n(t), \quad n = 1, 5, 10, 15, 20$

Question: What happens if we don't require that $t > 0$?

$$y_n(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_n(t)}{\tau_{n+1}(t)}, \qquad x_n(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\tau_n(t), \qquad \tau_n(t) = \left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}}\operatorname{Ai}(t)\right]_{j,k=0}^{n-1}$$



$y_n(t), \quad n = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$

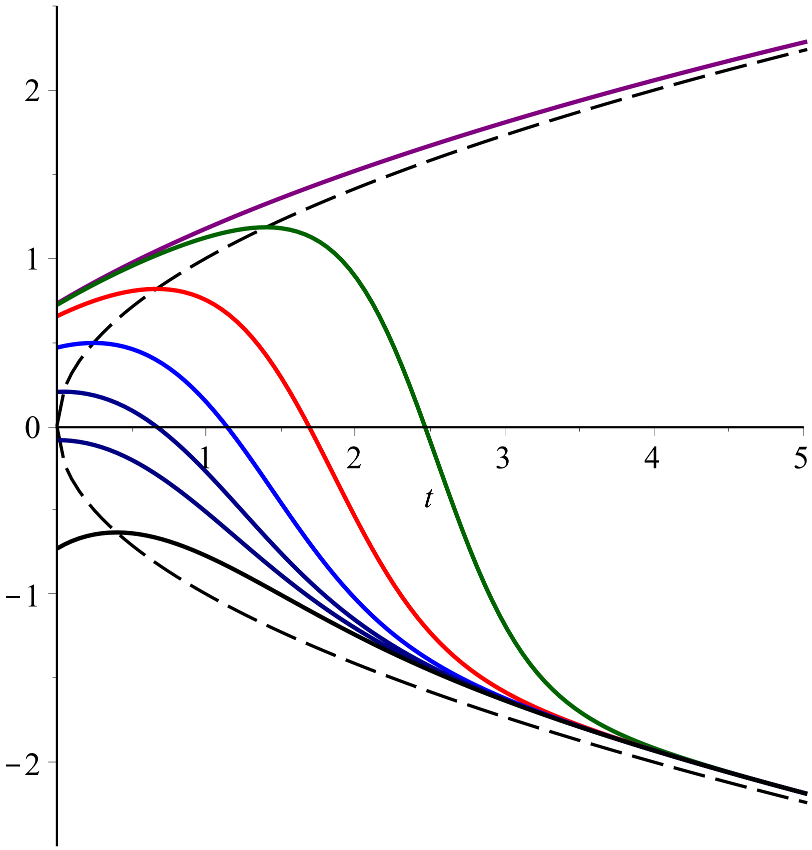


$\frac{1}{n}x_n(t), \quad n = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$

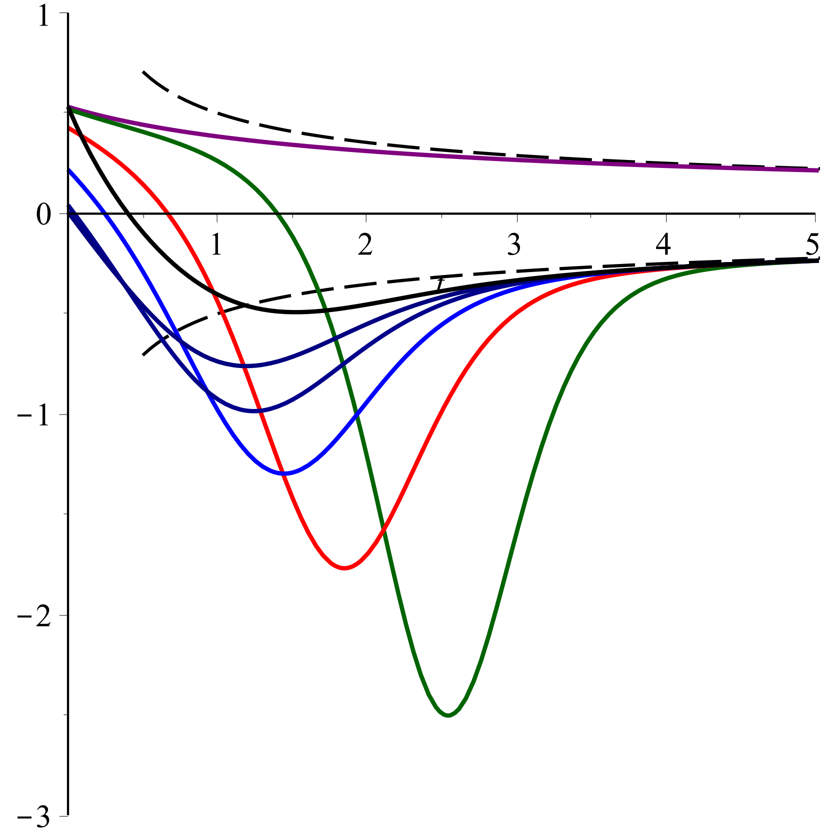
Question: What happens if we have a linear combination of $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$?

$$y_0(t;\vartheta) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln\varphi(t;\vartheta), \qquad x_1(t;\vartheta) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\varphi(t;\vartheta)$$

$$\varphi(t;\vartheta) = \cos(\vartheta)\,\mathrm{Ai}(t) + \sin(\vartheta)\,\mathrm{Bi}(t)$$



$y_0(t;\vartheta)$



$x_1(t;\vartheta)$

$$\vartheta = \mathbf{0},\ \tfrac{1}{1000}\boldsymbol{\pi},\ \tfrac{1}{100}\boldsymbol{\pi},\ \tfrac{1}{25}\boldsymbol{\pi},\ \tfrac{1}{10}\boldsymbol{\pi},\ \tfrac{1}{5}\boldsymbol{\pi},\ \tfrac{1}{2}\boldsymbol{\pi}$$

Airy Solutions of $\mathrm{P_{II}}$, $\mathrm{P_{34}}$ and $\mathrm{S_{II}}$

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + n + \tfrac{1}{2} \qquad \mathrm{P_{II}}$$

$$p\frac{\mathrm{d}^2 p}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^2 + 2p^3 - zp^2 - \tfrac{1}{2}n^2 \qquad \mathrm{P_{34}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}n^2 \qquad \mathrm{S_{II}}$$

- **PAC**, “On Airy Solutions of the Second Painlevé Equation”, *Studies in Applied Mathematics*, DOI: 10.1111/sapm.12123 (2016)

Airy Solutions of P_{II}, P_{34} and S_{II}

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}z^2} = 2q_n^3 + zq_n + n + \tfrac{1}{2} \qquad \mathrm{P}_{\mathrm{II}}$$

$$p_n\frac{\mathrm{d}^2 p_n}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p_n}{\mathrm{d}z}\right)^2 + 2p_n^3 - zp_n^2 - \tfrac{1}{2}n^2 \qquad \mathrm{P}_{34}$$

$$\left(\frac{\mathrm{d}^2\sigma_n}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma_n}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}n^2 \qquad \mathrm{S}_{\mathrm{II}}$$

Theorem

Let

$$\varphi(z;\vartheta) = \cos(\vartheta)\,\mathrm{Ai}(\zeta) + \sin(\vartheta)\,\mathrm{Bi}(\zeta), \qquad \zeta = -2^{-1/3}z$$

with ϑ an arbitrary constant, $\mathrm{Ai}(\zeta)$ and $\mathrm{Bi}(\zeta)$ Airy functions, and $\tau_n(z)$ be the Wronskian

$$\tau_n(z;\vartheta) = \mathcal{W}\left(\varphi, \frac{\mathrm{d}\varphi}{\mathrm{d}z}, \ldots, \frac{\mathrm{d}^{n-1}\varphi}{\mathrm{d}z^{n-1}}\right)$$

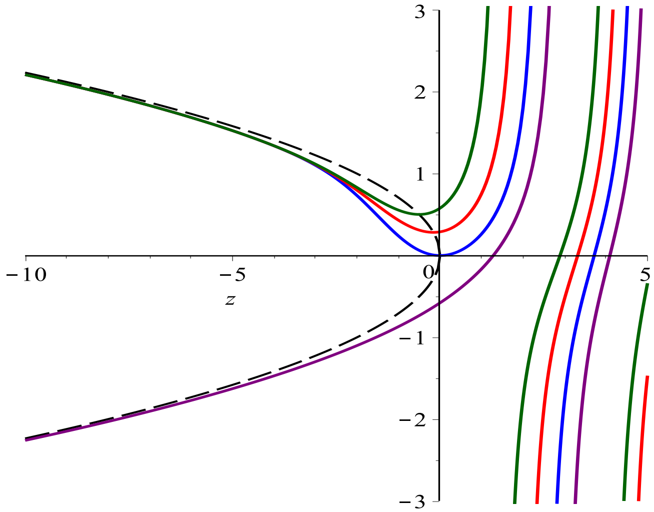
then

$$q_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_n(z;\vartheta)}{\tau_{n+1}(z;\vartheta)}, \quad p_n(z;\vartheta) = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;\vartheta), \quad \sigma_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z;\vartheta)$$

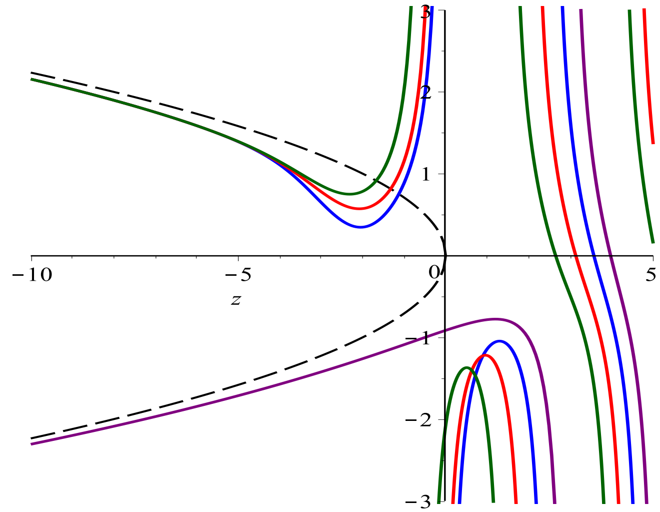
respectively satisfy P_{II}, P_{34} and S_{II}, with $n \in \mathbb{Z}$.

Airy Solutions of P_{II}

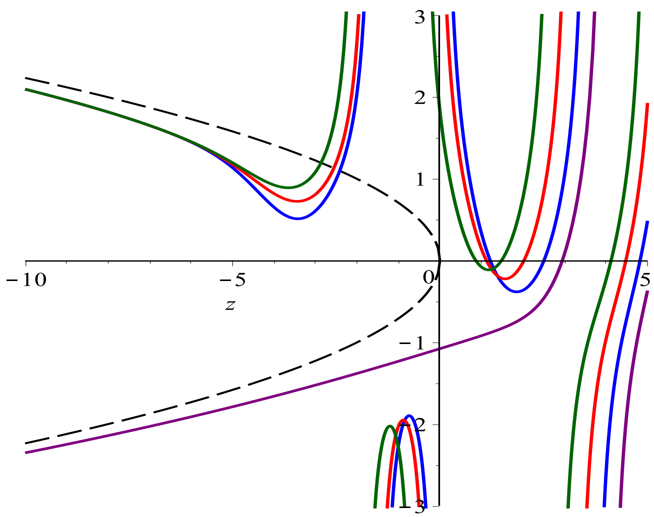
$$q_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{\tau_n(z;\vartheta)}{\tau_{n+1}(z;\vartheta)}$$



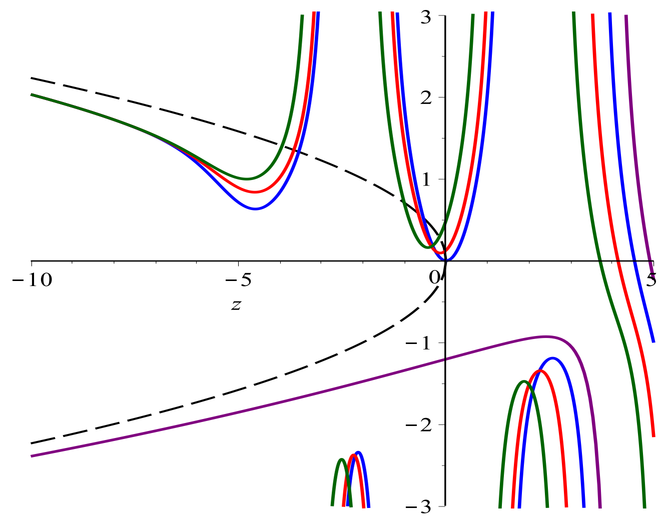
$n = 0, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$n = 1, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$n = 2, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$n = 3, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$

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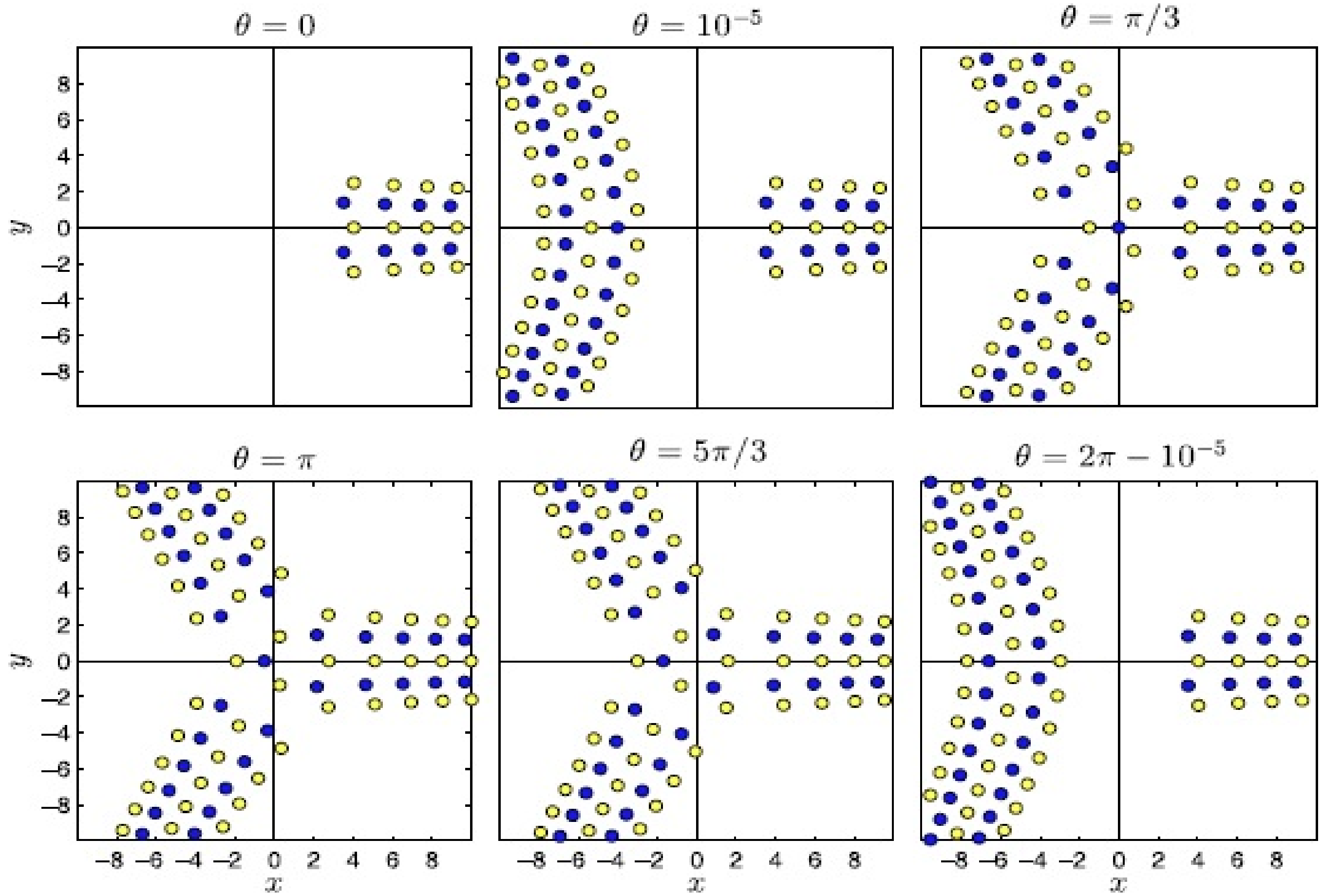
Airy Solutions of $\mathrm{P_{II}}$ with $\alpha = \frac{5}{2}$

(Fornberg & Weideman [2014])

$$q_2(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{\mathcal{W}(\varphi,\varphi')}{\mathcal{W}(\varphi,\varphi',\varphi'')}, \qquad \varphi(z;\vartheta) = \cos(\vartheta)\,\mathrm{Ai}(-2^{-1/3}z) + \sin(\vartheta)\,\mathrm{Bi}(-2^{-1/3}z)$$



blue/**yellow** denote poles with residue $+1/-1$

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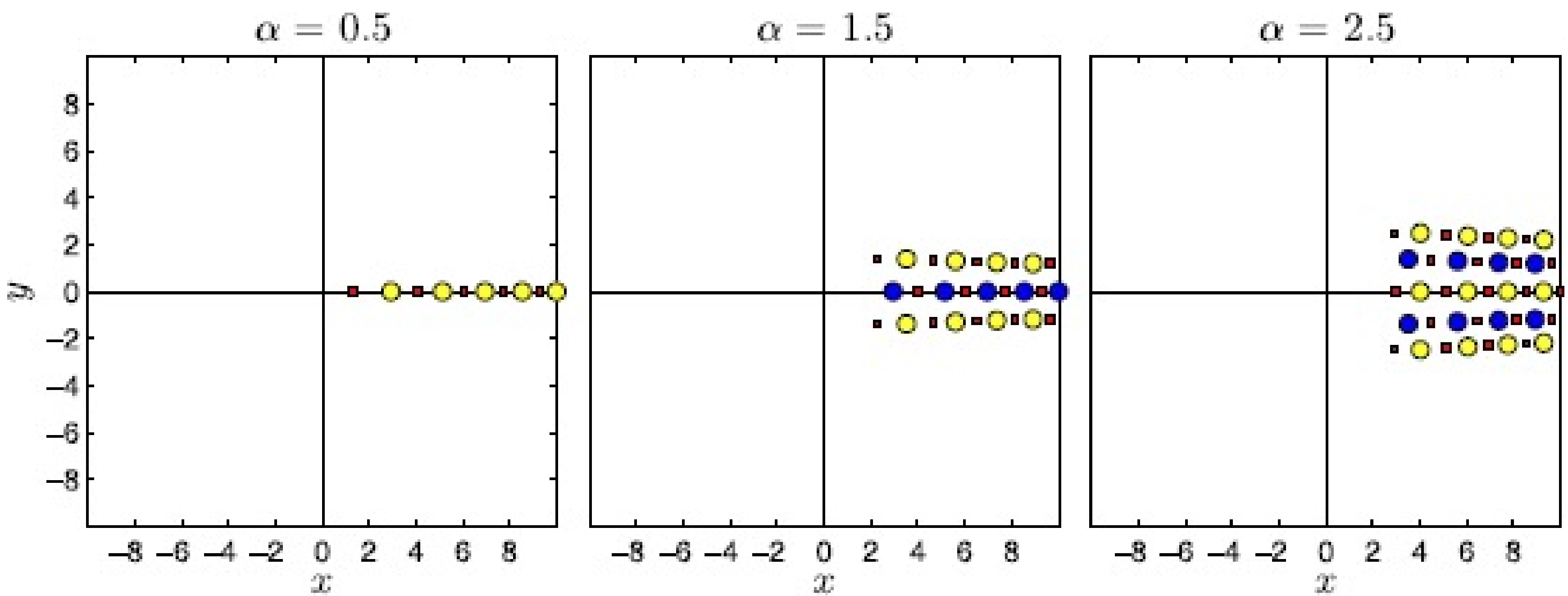
SANUM, Stellenbosch, March 2016

Tronquée Solutions of $\mathrm{P_{II}}$ (Airy with $\vartheta = 0$)

(Fornberg & Weideman [2014])

$$q_0(z;0) = -\frac{\mathrm{d}}{\mathrm{d}z}\ln\varphi, \quad q_1(z;0) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\mathcal{W}(\varphi)}{\mathcal{W}(\varphi,\varphi')}, \quad q_2(z;0) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\mathcal{W}(\varphi,\varphi')}{\mathcal{W}(\varphi,\varphi',\varphi'')}$$

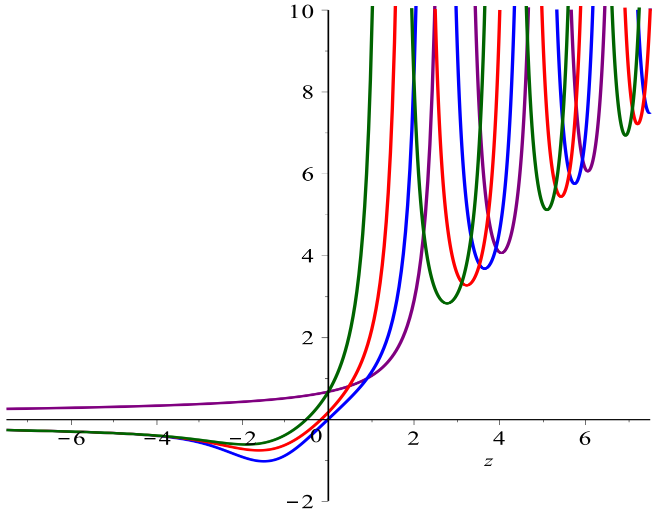
with $\varphi(z;0) = \mathrm{Ai}(-2^{-1/3}z)$



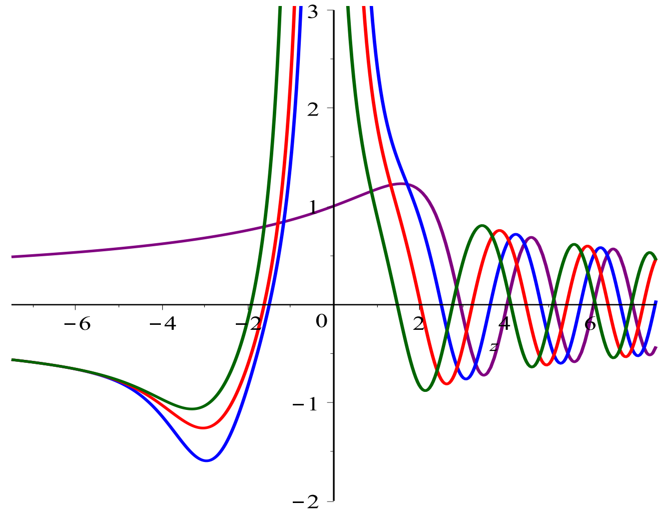
blue/**yellow** denote poles with residue $+1/-1$, **red** denote zeros

Airy Solutions of P_{34}

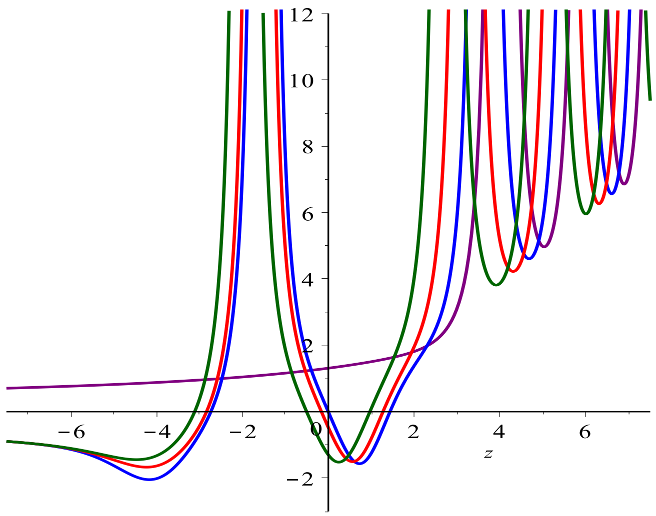
$$p_n(z;\vartheta) = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;\vartheta)$$



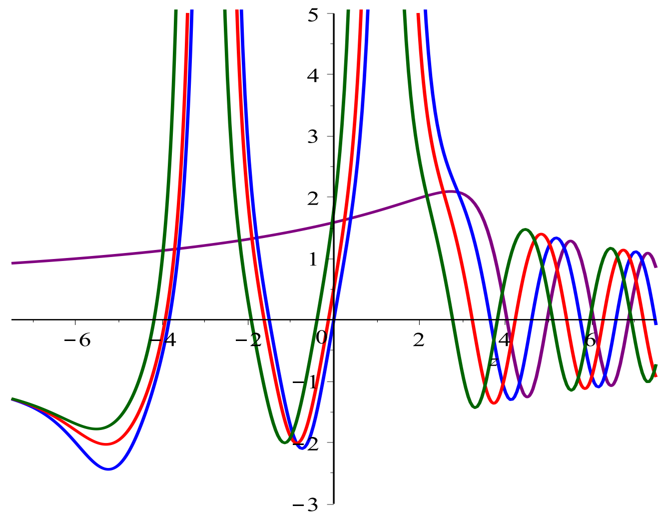
$n = 1, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$n = 2, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



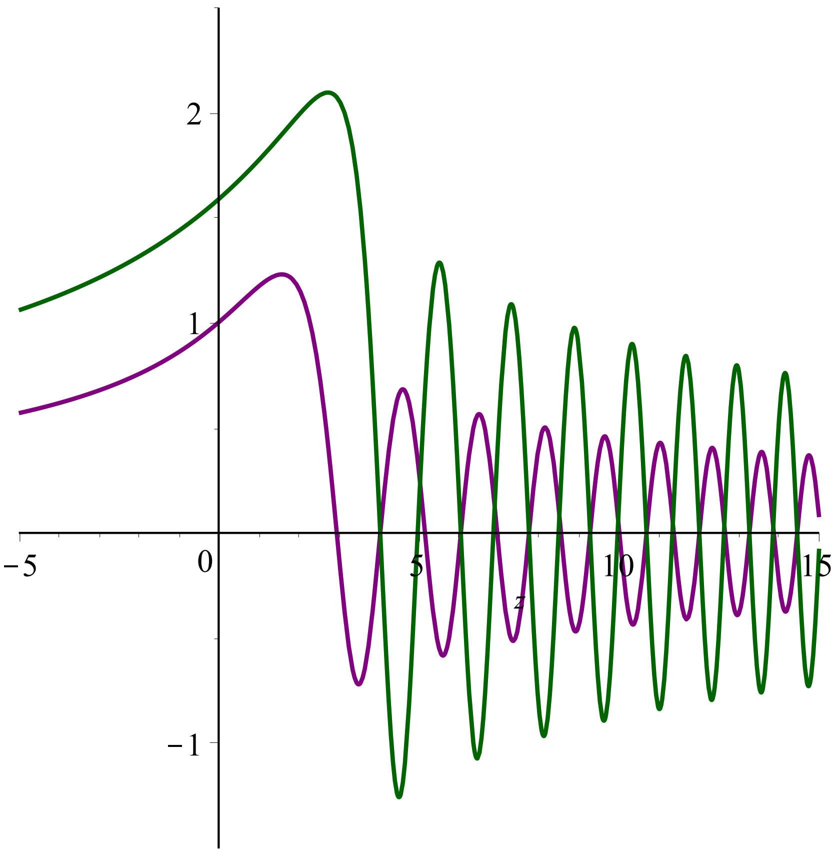
$n = 3, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



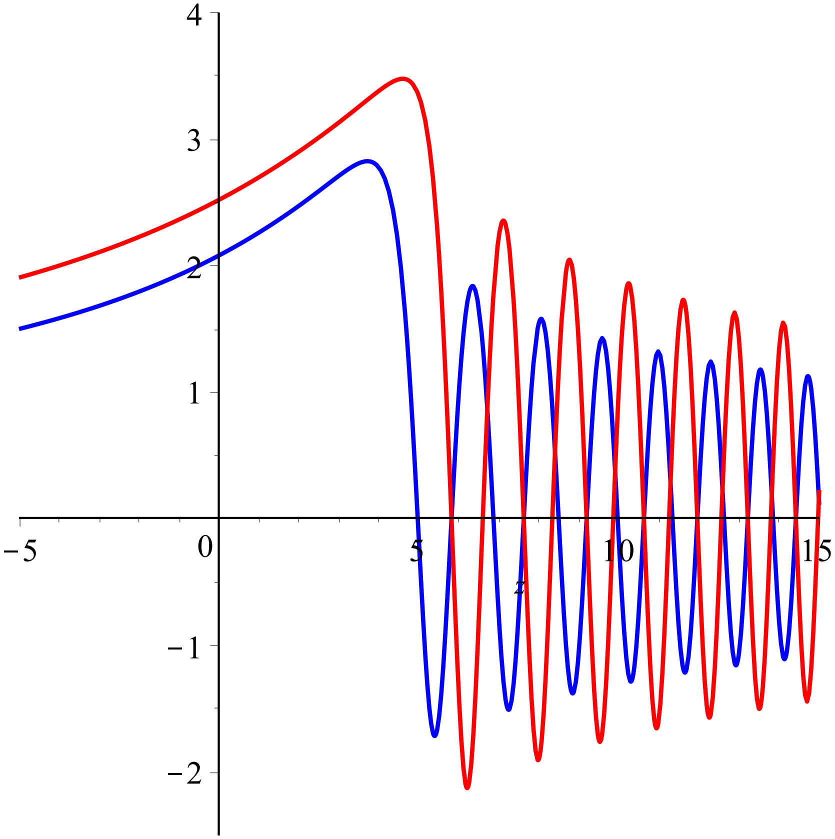
$n = 4, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$

Airy Solutions of P_{34}

$$p_n(z;0) = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;0)$$



$n = 2$, $n = 4$



$n = 6$, $n = 8$

Airy Solutions of P_{34}

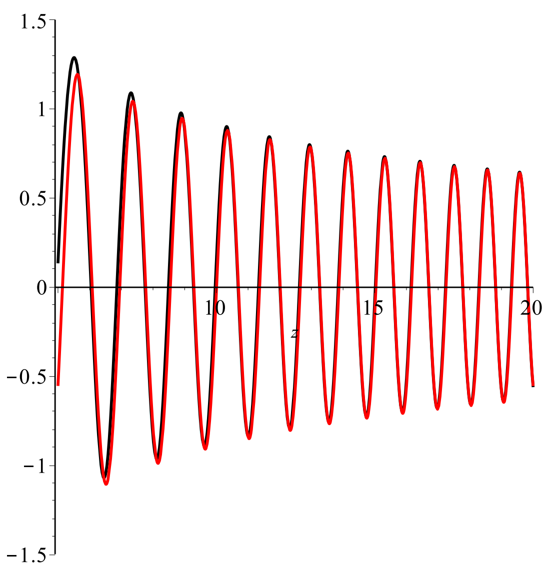
$$p_n\frac{\mathrm{d}^2p_n}{\mathrm{d}z^2}=\frac{1}{2}\left(\frac{\mathrm{d}p_n}{\mathrm{d}z}\right)^2+2p_n^3-zp_n^2-\tfrac{1}{2}n^2$$

Theorem

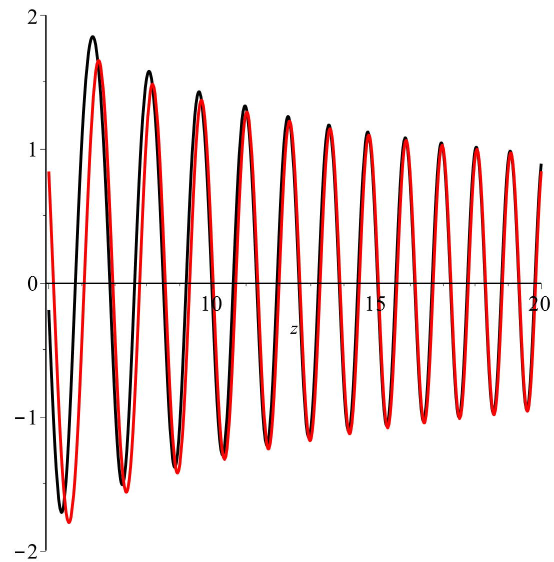
(**PAC [2016]**)

If $n\in 2\mathbb{Z}$***, then as*** $z\to\infty$

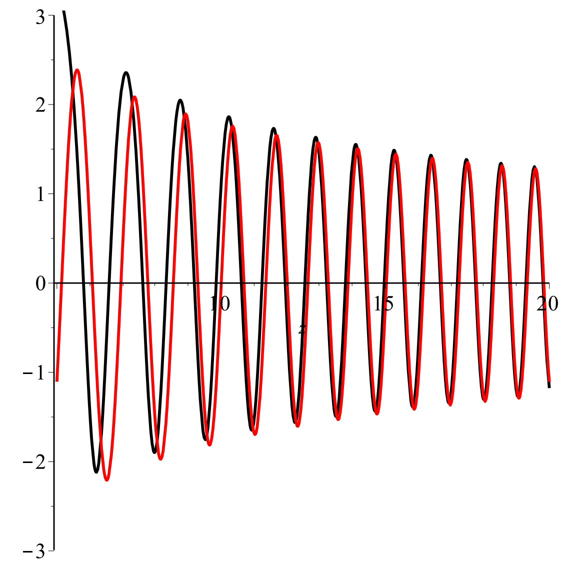
$$p_n(z;0)=\frac{n}{\sqrt{2z}}\cos\left(\tfrac{4}{3}\sqrt{2}\,z^{3/2}-\tfrac{1}{2}n\pi\right)+o\big(z^{-1/2}\big)$$



$n=4$



$n=6$



$n=8$

Its, Kuijlaars & Östensson [2008] discuss solutions of the equation

$$u_\beta \frac{\mathrm{d}^2 u_\beta}{\mathrm{d}t^2} = \tfrac{1}{2}\left(\frac{\mathrm{d}u_\beta}{\mathrm{d}t}\right)^2 + 4u_\beta^3 + 2tu_\beta^2 - 2\beta^2 \qquad (1)$$

where β is a constant, which is equivalent to P_{34} through the transformation

$$p(z) = 2^{1/3}u_\beta(t), \qquad t = -2^{-1/3}z,$$

and $\beta = \frac{1}{2}\alpha + \frac{1}{4}$ in their study of the double scaling limit of unitary random matrix ensembles.

Theorem (**Its, Kuijlaars & Östensson [2009]**)

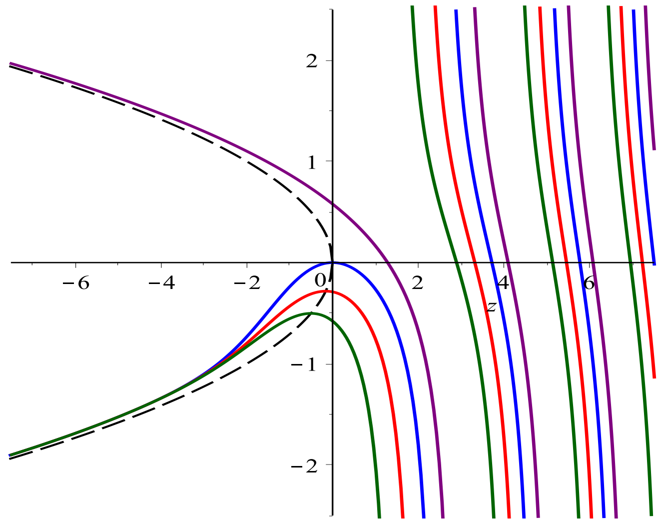
There are solutions $u_\beta(t)$ of (1) such that as $t \to \infty$

$$u_\beta(t) = \begin{cases} \beta t^{-1/2} + \mathcal{O}\left(t^{-2}\right), & \text{as} \quad t \to \infty \\ \beta(-t)^{-1/2}\cos\left\{\frac{4}{3}(-t)^{3/2} - \beta\pi\right\} + \mathcal{O}\left(t^{-2}\right), & \text{as} \quad t \to -\infty \end{cases} \qquad (2)$$

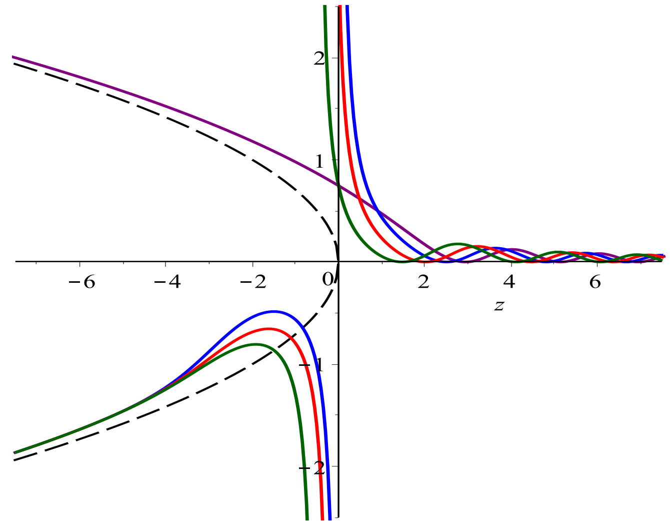
- Letting $\beta = 1$ in (2) shows that they are in agreement with the asymptotic expansions for $p_2(z;0)$.
- **Its, Kuijlaars & Östensson [2009]** conclude that solutions of (1) with asymptotic behaviour (2) are **tronquée solutions**, i.e. have no poles in a sector of the complex plane.

Airy Solutions of S_{II}

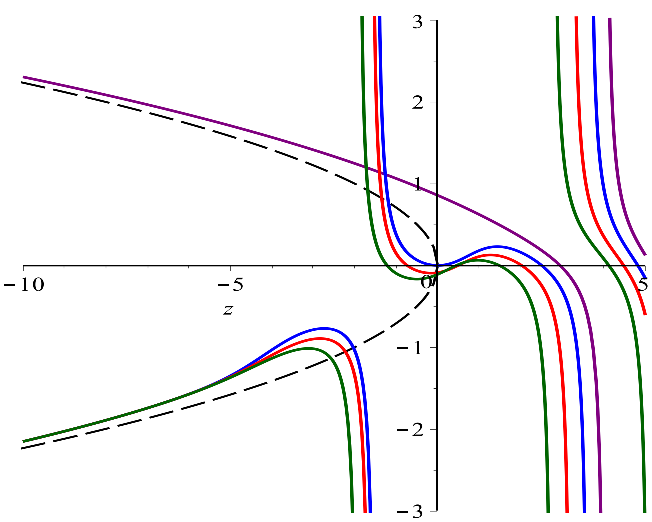
$$\sigma_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z;\vartheta)$$



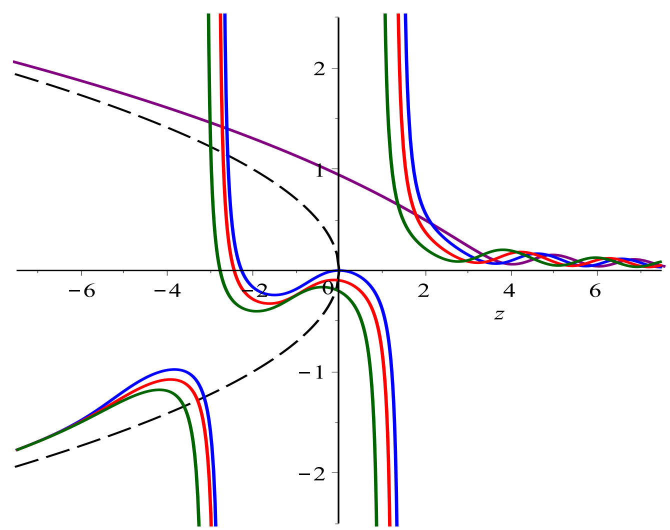
$\boldsymbol{n=1}, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n=2}, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n=3}, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n=4}, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$

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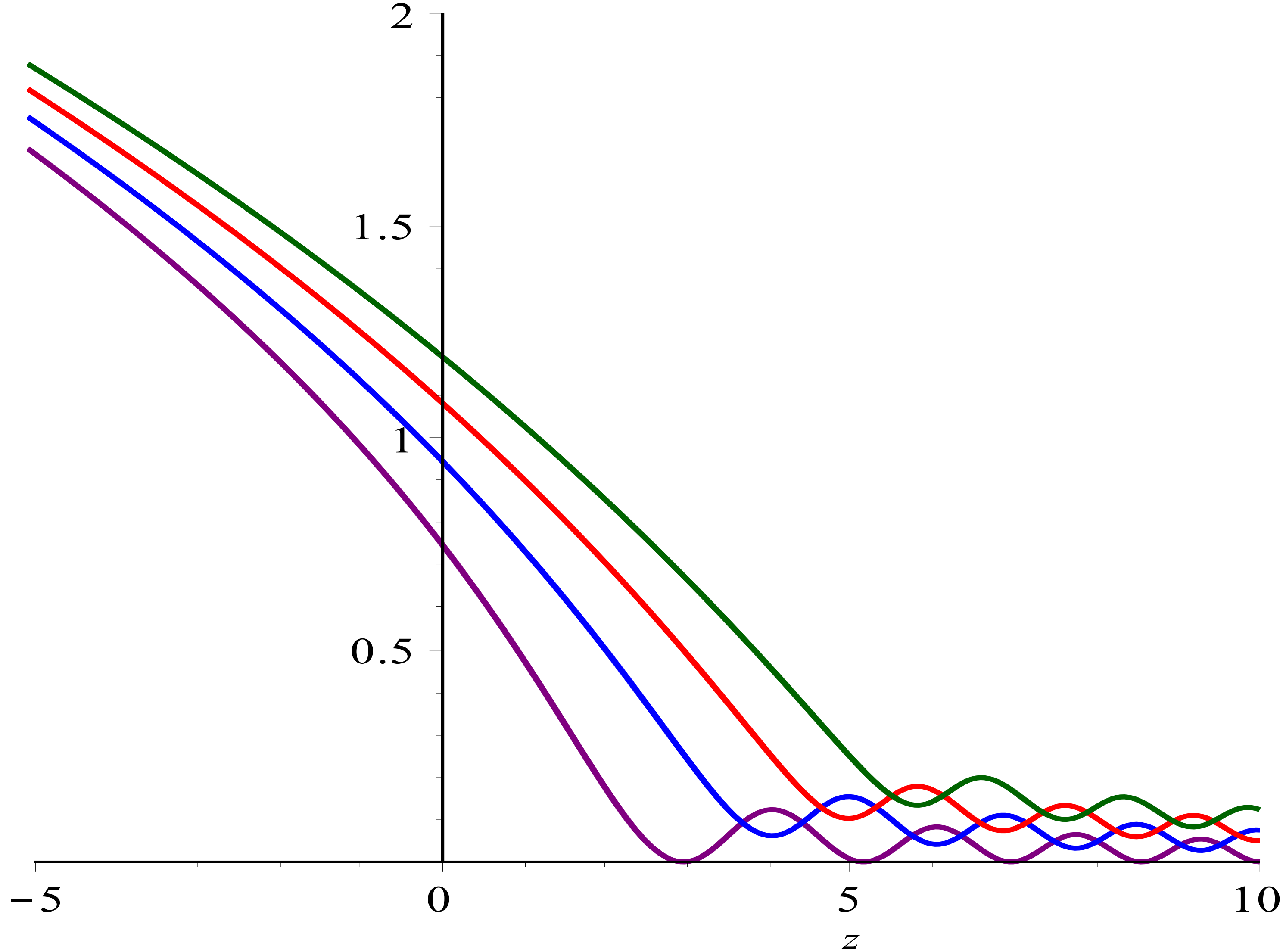
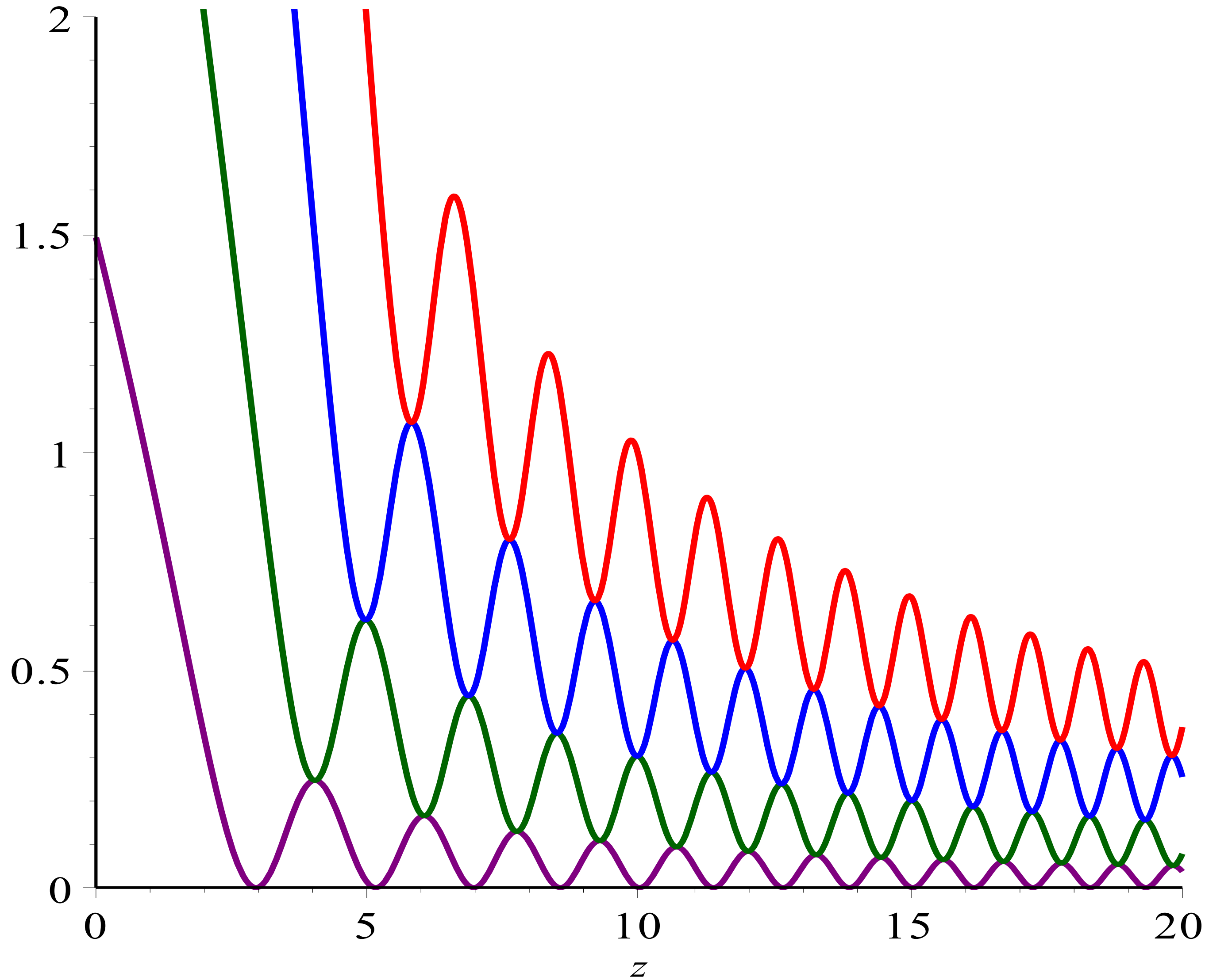
Airy Solutions of $\mathrm{S_{II}}$



Plots of $\sigma_n(z;0)/n$ for $n = \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{8}$

$$\sigma_n(z;0) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{W}\left(\varphi, \varphi', \ldots, \varphi^{(n-1)}\right), \qquad \varphi = \mathrm{Ai}(-2^{-1/3}z)$$



$n = 2$, $n = 4$, $n = 6$, $n = 8$

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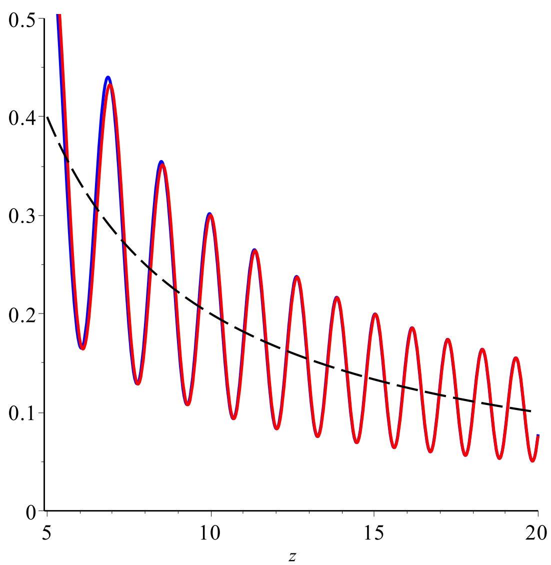
Airy Solutions of S_{II}

$$\left(\frac{\mathrm{d}^2\sigma_n}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma_n}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}n^2$$

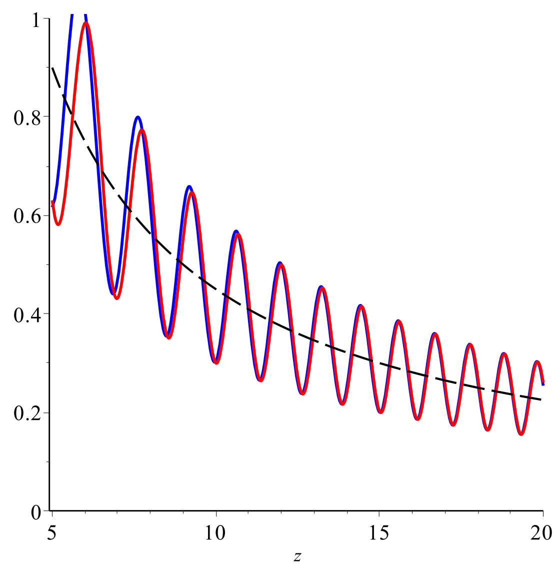
Theorem **(PAC [2016])**

If $n \in 2\mathbb{Z}$***, then as*** $z \to \infty$

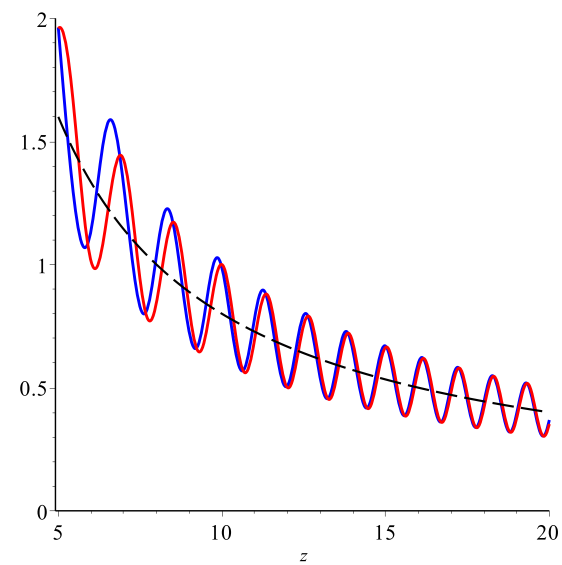
$$\sigma_n(z;0) = \frac{n}{8z}\left\{n - 2\sin\left(\tfrac{4}{3}\sqrt{2}\,z^{3/2} - \tfrac{1}{2}n\pi\right)\right\} + o\left(z^{-1}\right)$$



$n = 4$



$n = 6$



$n = 8$

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14th International Symposium on "*Orthogonal Polynomials, Special Functions and Applications*"

University of Kent, Canterbury, UK

3rd-7th July 2017

7th Summer School on "*Orthogonal Polynomials and Special Functions*"

University of Kent, Canterbury, UK

26th-30th June 2017

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