Laguerre Polynomials and Interlacing of Zeros

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SANUM Conference

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Joint work with Martin Muldoon

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Overview of Talk

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Sharpness of t- interval where zeros of Laguerre polynomials $L_n^{(\alpha)}$ and $L_{n-k}^{(\alpha+t)}$ are interlacing. $\alpha > -1, t > 0$. Askey Conjecture

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Sharpness of t- interval where zeros of Laguerre polynomials $L_n^{(\alpha)}$ and $L_{n-k}^{(\alpha+t)}$ are interlacing. $\alpha > -1, t > 0$. Askey Conjecture

Breakdown of interlacing of zeros of $L_n^{(\alpha)}$ and $L_{n-1}^{(\alpha)}$ when $-2 < \alpha < -1$. Add one point to restore interlacing. Quasi-orthogonal order 1 case.

Laguerre Polynomial $L_n^{(\alpha)}$

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Laguerre polynomial $L_n^{(\alpha)}$ defined by

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For $\alpha > -1$, sequence $\{L_n^{(\alpha)}(x),\ n = 0, 1, \dots\}$ orthogonal with respect to $x^\alpha e^{-x}$ on $(0, \infty)$. All n zeros of L_n^α are real, distinct and positive.

$$\int_0^\infty L_n(x) L_m(x) x^\alpha e^{-x}\, dx = 0 \quad \text{for} \quad n \neq m,$$

$$\int_0^\infty L_n L_n x^\alpha e^{-x}\, dx \neq 0$$

$\alpha > -1$ necessary for convergence of integral each m, n.

Interlacing of zeros

Orthogonal sequence $\{p_n\}_{n=0}^{\infty}$, zeros of p_n and p_{n-1} are interlacing:

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p, q real polynomials, real, simple, disjoint zeros,deg(p)$>$ deg(q), zeros of p and q interlace if each zero of q lies between two successive zeros of p and at most one zero of q between any two successive zeros of p.

Askey Conjecture 1989

Zeros of orthogonal Jacobi polynomials $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha+2,\beta)}$ interlace

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Askey Conjecture. Driver, Jordaan and Mbuyi: Zeros of Jacobi $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha+k,\beta-l)}$ interlace if $k, l \leq 2$ provided $\beta - l$ remains > -1.

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Ismail, Dimitrov and Rafaeli: Askey Conjecture sharp.

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Conjecture Kerstin Jordaan -KD 2011

Zeros L_n^α and $L_{n-2}^{\alpha+t}$ interlace for $0 \le t \le 4$ if the two polynomials have no common zeros

Sharp interval. Zeros of Laguerre polynomials

Martin Muldoon and KD Journal of Approximation Theory 2013

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Interval $0 \leq t \leq 2k$ is largest possible for which interlacing holds each n, α, k

Proofs of interlacing and sharpness results

t interval $0 \leq t \leq 2k$ largest possible for interlacing of zeros of $L_n^{(\alpha)}$, $L_{n-k}^{(\alpha+t)}$, each $n \in \mathbb{N}$, $0 < k \leq n-2$, and fixed $\alpha \geq 0$, (excluding t for which common zeros occur)

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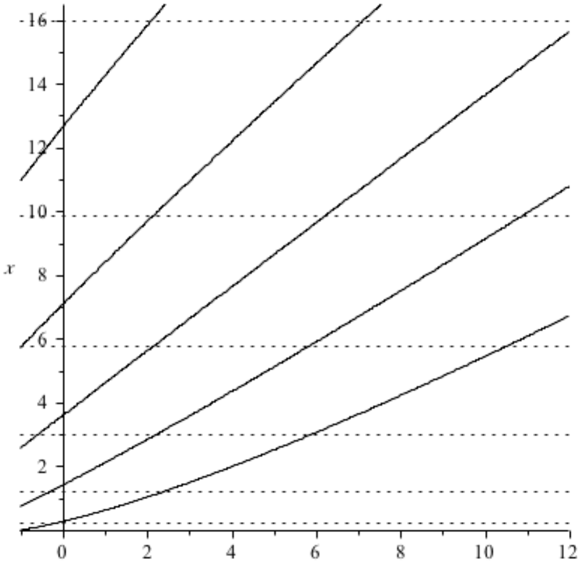
Sharpness

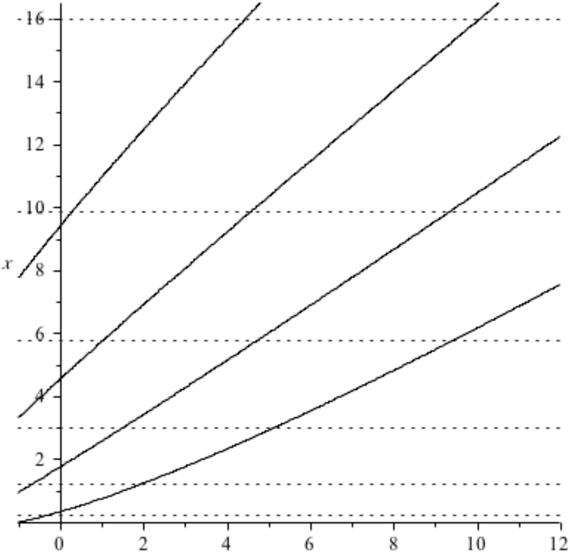
Inequalities satisfied by zeros of $L_n^{(\alpha)}$ and zeros of Bessel function

Asymptotic behaviour of zeros of $L_n^{(\alpha)}$ using Airy function

Zeros of $L_6^0(x)$ and $L_5^t(x)$ as a function of t



Zeros of $L_6^0(x)$ and $L_4^t(x)$ as a function of t



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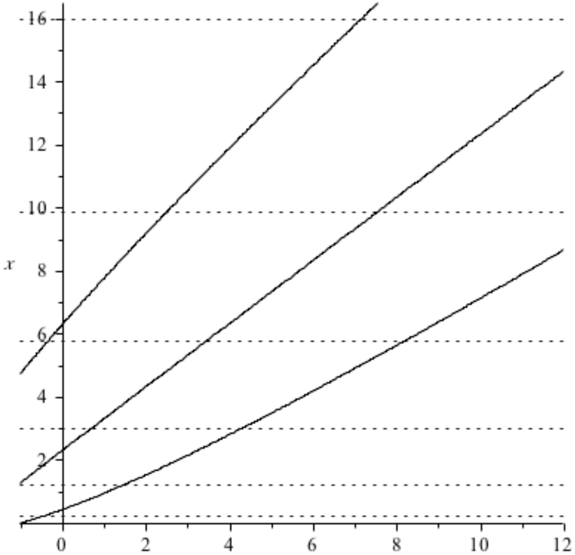

Laguerre sequences $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$, $-2 < \alpha < -1$

As α decreases below -1, one positive zero of $L_n^{(\alpha)}$ leaves $(0, \infty)$ each time α passes through $-1, -2, \ldots, -n$.

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Brezinski, Driver, Redivo-Zaglia 2004

If $-2 < \alpha < -1$, zeros of $L_n^{(\alpha)}$ are real, simple, $n-1$ are positive, 1 negative

Zeros of Laguerre polynomials $L_n^{(\alpha)}$, $-2 < \alpha < -1$

Question Muldoon-Driver

Interlacing of zeros of $L_n^{(\alpha)}$ and zeros of $L_{n-1}^{(\alpha)}$ when $-2 < \alpha < -1$??

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Sequence not orthogonal but all zeros real–is there interlacing for any $n \in \mathbb{N}$?

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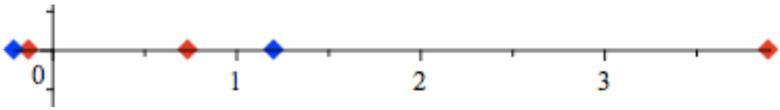
Negative zero of $L_n^{(\alpha)}$ increases with n (Ismail, Zeng 2015)

Zeros of $L_2^{(-3/2)}$ (blue) and $L_3^{(-3/2)}$ (red)



Thank you