Fluid driven hydraulic fracture in a permeable medium

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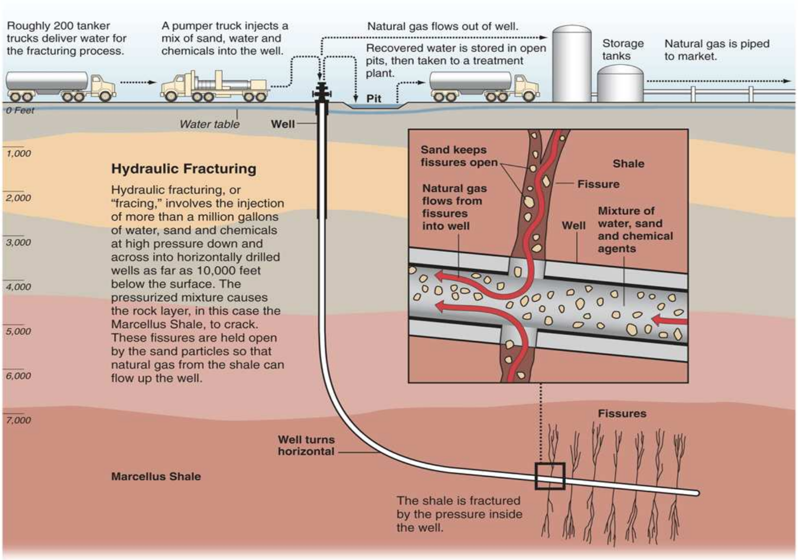
Introduction

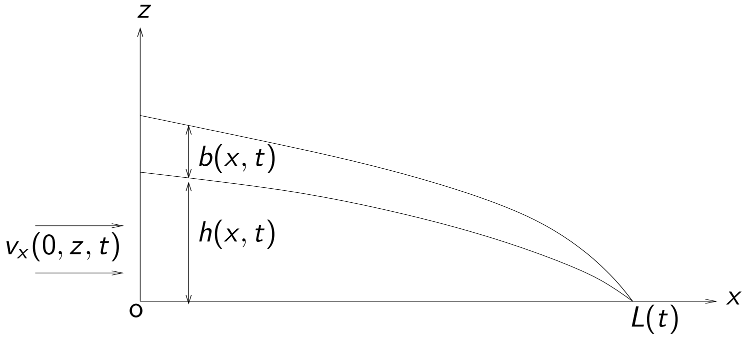
- Hydraulic fracturing(also called Fracking) is the process by which fractures in rocks are propagated by the injection of high pressure viscous fluid into the fracture
- Hydraulic fracture technique is a core technology in the production of petroleum, natural gas, natural gas liquids such as ethane and propane trapped within rock layer thousands of feet($>$ 2000metres) below the earth surface



Graphic by Al Granberg

Mathematical formulation

A two-dimensional fracture driven by an incompressible Newtonian fluid.



Model Assumptions

The following assumptions are made for our model:

- The injected fluid is Newtonian and fluid flow inside the fracture is laminar
- The rock is a permeable medium and there is fluid leak-off into the rock matrix.
- The rock is a linearly elastic material which assumes small displacement gradients.
- The fracture propagates along the positive x-direction, is one-sided, $0 \leq x \leq L(t)$, identical in every plane y=constant and has length $L(t)$ and half-width $h(x,t)$.
- The flow of fluid inside the fracture is modelled using lubrication theory.

Fluid flow equations in the fracture

$$\nabla.\vec{v} = 0$$

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}.\nabla)\vec{v} = -\nabla p + \mu\nabla^2\vec{v}$$

Fluid flow equations in the porous matrix

$$\frac{Q}{A} = \frac{\partial(b+h)}{\partial t} = -\frac{\kappa}{\mu}\nabla p_d$$

$p(x,t)$ is the fluid pressure, ρ is the fluid density

μ is the fluid viscosity, $\frac{Q}{A}$ is the volume flow per unit area

κ is the permeability

Body force is neglected

By making the thin fluid film approximation of lubrication theory,

$$\epsilon = \frac{H}{L_0} << 1, \quad \epsilon^2 Re << 1,$$

where

- L_0 is a typical fracture length,
- T is characteristic time it takes to initiate fracture. If there is fluid leak-off, $T > \frac{L}{U}$ (N.N Smirnov and V.R Tagirova)
- H is a typical fracture half-width,
- U is a typical fluid speed in the x-direction and
- Re, the Reynolds number is $\frac{\rho U L_0}{\mu}$
- the characteristic pressure is defined as $\frac{\mu U}{L_0 \varepsilon^2}$,

Two-dimensional lubrication theory equations in dimensional form:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 v_x}{\partial z^2}, \quad \frac{\partial p}{\partial z} = 0, \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0.$$

Darcy equation

$$\frac{\partial b}{\partial t} = -\frac{\kappa}{\mu} \frac{\partial p_d}{\partial z}$$

Boundary conditions and PKN approximation

$$z = h(x,t): \ \ v_x(x, h(x,t), t) = 0,$$

$$z = h(x,t): \ \ v_z(x, h(x,t), t) = \frac{\partial(h+b)}{\partial t}.$$

$$z = 0: \quad v_z(x,0,t) = 0, \quad \frac{\partial v_x}{\partial z}(x,0,t) = 0.$$

$$p = p_f - \sigma_0 = \Lambda h, \ \text{where } \Lambda = \frac{E}{(1-\sigma^2)B}$$

E and σ are Youngs modulus and Poisson ratio respectively and B is the unit breadth along y.

$$p_d(x, h+b, t) = 0 \text{ and } p_d(x,h,t) = \Lambda h$$

Flow velocity:

$$v_x = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(z^2 - h^2\right)$$

Nonlinear equations

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h\bar{v}_x\right) = -\frac{\partial b}{\partial t},$$

$$\frac{\partial b}{\partial t} = \frac{\Lambda\kappa}{\mu}\frac{h}{b}$$

where

$$\bar{v}_x = -\frac{h^2}{3\mu}\frac{\partial p}{\partial x}$$

Dimensionless equations

$$\Omega \frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) + \frac{1}{\Gamma}\frac{h}{b} = 0$$

$$\frac{\partial b}{\partial t} = \frac{h}{b}$$

At the fracture tip, $x = L(t)$:

$$h(L(t), t) = 0, \text{ and } b(L(t), t) = 0.$$

The initial conditions are

$$t = 0 : \qquad L(0) = 1, \qquad h(0,0) = 1.$$

A pre-existing fracture exists in the rock mass:

$$t = 0 : \qquad h(0, x) = h_0(x), \qquad 0 \leq x \leq L(t),$$

where $h_0(0) = 1$. Dimensionless numbers: $\Omega = \frac{LH}{UTH}$ and $\Gamma = \frac{UH}{v_l L}$

Global mass balance

$$\left(\begin{array}{c}\text{rate of change of total}\\ \text{volume of fracture}\end{array}\right) = \left(\begin{array}{c}\text{rate of flow of fluid into}\\ \text{fracture at the fracture entry}\end{array}\right)$$
$$- \left(\begin{array}{c}\text{rate of flow of leaked-off}\\ \text{fluid at the fluid-rock interface}\end{array}\right).$$

That is,

$$\frac{dV}{dt} = Q_1 - Q_2,$$

where

$$V(t) = 2\int_0^{L(t)} h(x,t)\,\mathrm{d}x,$$

$$Q_1(0,t) = 2\int_0^{h(0,t)} v_x(0,z,t)\,\mathrm{d}z = 2h(0,t)\bar{v}_x(0,t),$$

and

$$Q_2(t) = 2\int_0^{L(t)} \frac{\partial b}{\partial t}(x,t)\,\mathrm{d}x.$$

The problem is therefore to solve the nonlinear diffusion equation

$$\Omega \frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) + \frac{1}{\Gamma}\frac{h}{b} = 0$$

$$\frac{\partial b}{\partial t} = \frac{h}{b}$$

for the fracture half-width subject to the boundary condition

$$h(L(t), t) = 0 \text{ and } b(L(t), t).$$

and the balance law

$$\frac{dV}{dt} = -2h^3(0,t)\frac{\partial h}{\partial x}(0,t) - 2\int_0^{L(t)} \frac{\partial b}{\partial t}(x,t)\,\mathrm{d}x,$$

where $\Omega = \frac{LH}{UTH}$ and $\Gamma = \frac{UH}{v_l L}$

$h = \Phi(x,t)$ and $b = \Psi(x,t)$ are group invariant solutions provided

$$X\left(h - \Phi(x,t)\right)\Big|_{h=\Phi} = 0.$$

$$X\left(b - \Psi(x,t)\right)\Big|_{b=\Psi} = 0.$$

where

$$X = (c_1 + c_2 t)\frac{\partial}{\partial t} + (c_4 + 2c_2 x)\frac{\partial}{\partial x} + c_2 h\frac{\partial}{\partial h} + c_2 b\frac{\partial}{\partial b}$$

$$(c_1 + c_2 t)\frac{\partial h}{\partial t} + (c_4 + 2c_2 x)\frac{\partial h}{\partial x} = c_2 h$$

$$(c_1 + c_2 t)\frac{\partial b}{\partial t} + (c_4 + 2c_2 x)\frac{\partial b}{\partial x} = c_2 b$$

The Case $c_2 = 0$ yields solution of the traveling waves type.

$$h(x,t) = f(\xi), \qquad b(x,t) = g(\xi)$$

where $\xi = x - \frac{c_4}{c_1}t$

Case $c_2 \neq 0$

Group invariant solution for the half-width and leak-off depth:

$$h(x,t) = (c_1 + c_2 t)\, f(\xi) \;\text{ and }\; b(x,t) = (c_1 + c_2 t)\, g(\xi)$$

where

$$\xi = \frac{c_4 + 2c_2 x}{(c_1 + c_2 t)^2}$$

Choose $c_4 = 0$ so that $\xi = 0$ when $x = 0$.

Boundary condition $h(L(t), t)) = 0$ implies $f(w) = 0$, where

$$w(t) = \frac{2c_2 L(t)}{(c_1 + c_2 t)^2}$$

$$\frac{df}{dw}\frac{dw}{dt} = 0 \Longrightarrow L(t) = \left(1 + \frac{c_2}{c_1} t\right)^2$$

$$u = \frac{x}{L(t)}, \quad \xi = \frac{2c_2}{c_1^2} u, \quad f(\xi) = \left(\frac{c_2}{c_1^4}\right)^{\frac{1}{3}} F(u), \quad g(\xi) = \frac{1}{c_2} G(u)$$

Since $h(0,0) = 1$, $f(0) = \frac{1}{c_2}$ and $F(0) = \left(\frac{c_2}{c_1}\right)^{-\frac{1}{3}}$

The problem is to solve the system

$$\Omega\left(F(u) - 2u\frac{dF}{du}\right) - \frac{d}{du}\left(F^3(u)\frac{dF}{du}\right) + \frac{1}{\Gamma}\frac{F(u)}{G(u)} = 0$$

$$2u\frac{dG}{du} - G(u) + \left(\frac{c_2}{c_1}\right)^{\frac{4}{3}}\frac{F(u)}{G(u)} = 0$$

subject to the boundary conditions

$$F(1) = 0, \quad G(1) = 0$$

$$F(0)^3\frac{dF}{du}(0) = -3\left[\int_0^1 F(u)\mathrm{d}u + \left(\frac{c_2}{c_1}\right)^{-\frac{4}{3}}\int_0^1 G(u)\mathrm{d}u\right].$$

where $F(0) = \left(\frac{c_2}{c_1}\right)^{-\frac{1}{3}}$

Once $F(u)$ has been calculated, $h(x,t)$ and $b(x,t)$ are obtained from

$$h(x,t) = \left(1 + \frac{c_2}{c_1}t\right)\frac{F(u)}{F(0)},$$

$$b(x,t) = \left(1 + \frac{c_2}{c_1}t\right)F(0)^3 G(u),$$

Case $\Omega = \frac{LH}{UTH} \ll 1$ and $\Gamma = \frac{UH}{v_l L} \sim 1$ (Strong leak-off)

$$\frac{d}{du}\left(F^3(u)\frac{dF}{du}\right) - \frac{F(u)}{G(u)} = 0$$

$$2u\frac{dG}{du} - G(u) + \left(\frac{c_2}{c_1}\right)^{\frac{4}{3}} \frac{F(u)}{G(u)} = 0$$

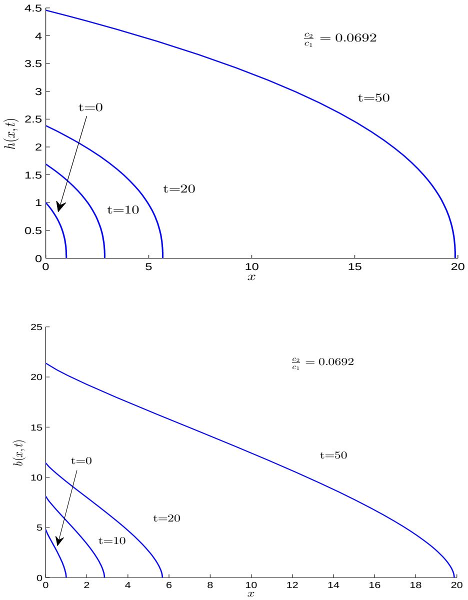
$$\text{subject to} \quad F(1) = 0, \quad G(1) = 0$$

$$F(0)^3\frac{dF}{du}(0) = -3\left[\int_0^1 F(u)\mathrm{d}u + \left(\frac{c_2}{c_1}\right)^{-\frac{4}{3}} \int_0^1 G(u)\mathrm{d}u\right].$$

where $F(0) = \left(\frac{c_2}{c_1}\right)^{-\frac{1}{3}}$.

$F(u) \sim \left(\frac{686}{45}\right)^{\frac{1}{7}} \left(\frac{c_2}{c_1}\right)^{-\frac{4}{21}} (1-u)^{\frac{3}{7}}$

$G(u) \sim \frac{49}{15}\left(\frac{45}{686}\right)^{\frac{3}{7}} \left(\frac{c_2}{c_1}\right)^{\frac{12}{21}} (1-u)^{\frac{5}{7}}$



$\frac{c_2}{c_1} = 0.0692$
t=0
t=10
t=20
t=50
$h(x,t)$
x
$\frac{c_2}{c_1} = 0.0692$
t=0
t=10
t=20
t=50
$b(x,t)$
x