A computational study of a class of multivalued tronquée solutions of the third Painlevé equation

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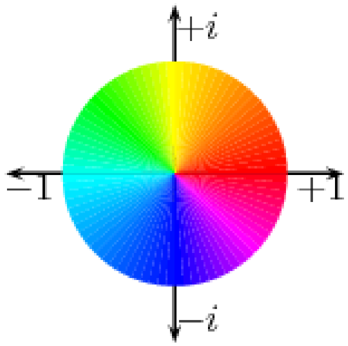
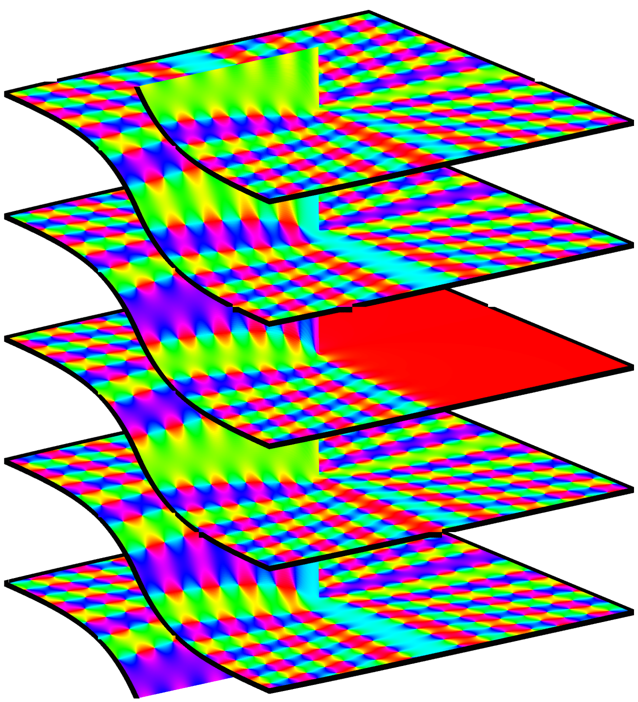
Introducing P_{III}

$$P_{III}: \quad \frac{d^2u}{dz^2} = \frac{1}{u}\left(\frac{du}{dz}\right)^2 - \frac{1}{z}\frac{du}{dz} + \frac{\boldsymbol{\alpha} u^2 + \boldsymbol{\beta}}{z} + \boldsymbol{\gamma} u^3 + \frac{\boldsymbol{\delta}}{u}$$

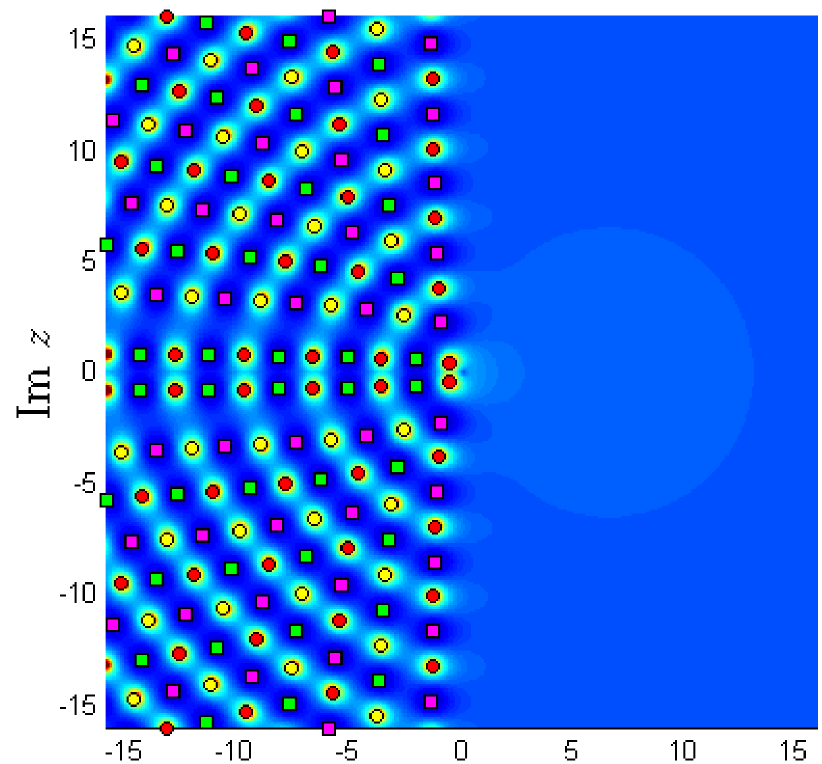
Phase portrait

$$u(z) = r(z)e^{i\theta(z)}$$





Modulus plot



- **Poles:** $u \sim \frac{c}{z - z_0}, \quad z \longrightarrow z_0$

  $c = \pm 1$, (red/yellow)

- **Zeros:** $u \sim c(z - z_0), \quad z \longrightarrow z_0$

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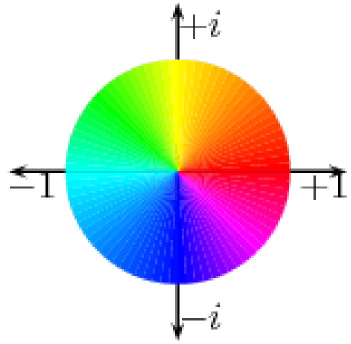

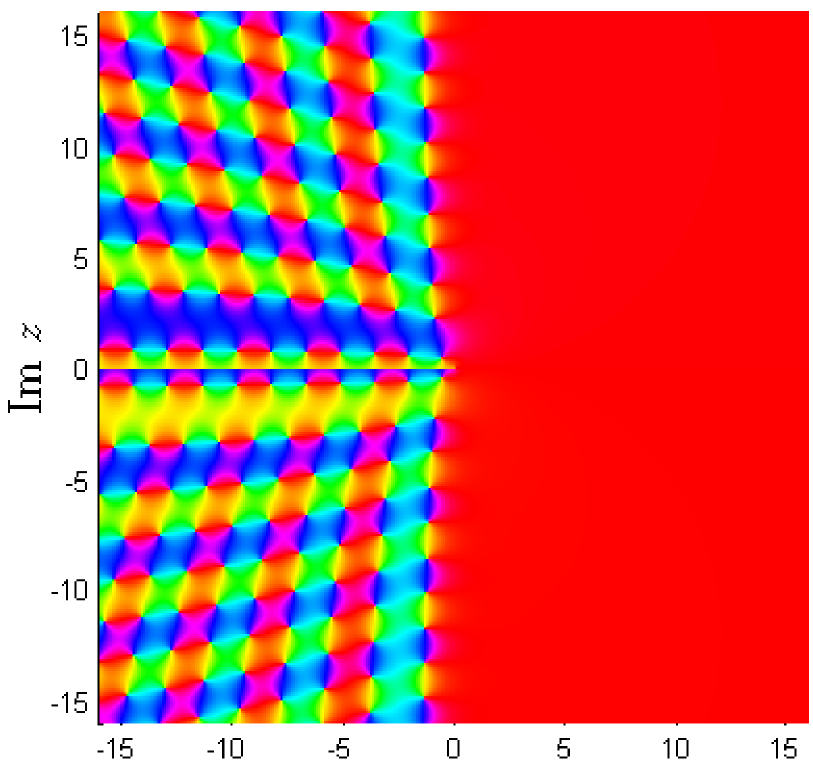

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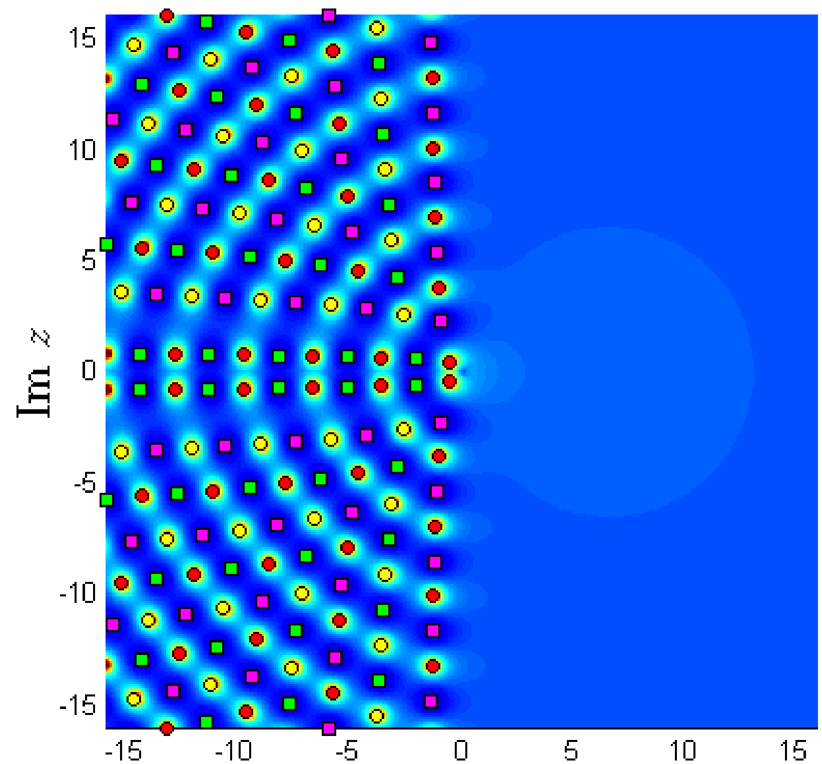

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Tronquée P_{III} solutions

Tronquée solutions, **Boutroux [1913]**

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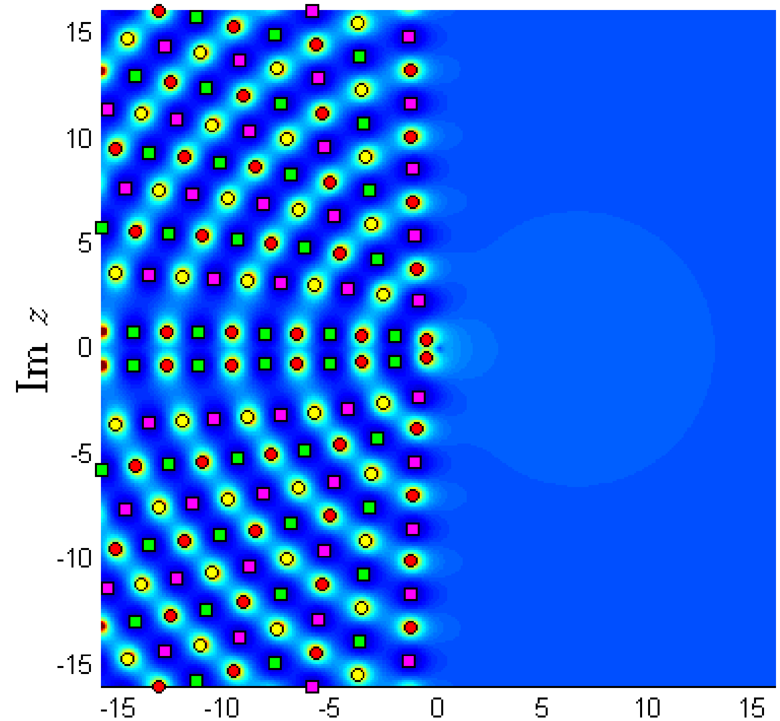

Lin, Dai and Tibboel [2014] proved the existence of tronquée P_{III} solutions whose pole-free sectors have angular widths of

- π and 2π if $\gamma = 1$ and $\delta = -1$, and
- $\frac{3\pi}{2}$ and 3π if $\alpha = 1, \gamma = 0$ and $\delta = -1$.

McCoy, Tracy and Wu [1977] derived asymptotic formulae for tronquée solutions with parameters $\alpha = -\beta = 2\nu$, $\gamma = 1$ and $\delta = -1$ and applied their results to the Ising model.

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$$u(x)\ \sim\ 1-\lambda\,\Gamma\left(\nu+\tfrac{1}{2}\right)2^{-2\nu}x^{-\nu-(1/2)}e^{-2x},\qquad x\to\infty$$

$$u(x)\ \sim\ Bx^{\sigma},\qquad x\to 0^+$$

$$\sigma(\lambda)\ =\ \frac{2}{\pi}\arcsin(\pi\lambda)\,,$$

$$B(\sigma,\nu)\ =\ 2^{-2\sigma}\frac{\Gamma^2\left(\frac{1}{2}(1-\sigma)\right)\Gamma\left(\frac{1}{2}(1+\sigma)+\nu\right)}{\Gamma^2\left(\frac{1}{2}(1+\sigma)\right)\Gamma\left(\frac{1}{2}(1-\sigma)+\nu\right)}$$

$$-\frac{1}{\pi}<\ \lambda\ <\frac{1}{\pi}$$

$$u(x;\nu,-\lambda)\ =\ \frac{1}{u(x;\nu,\lambda)}$$

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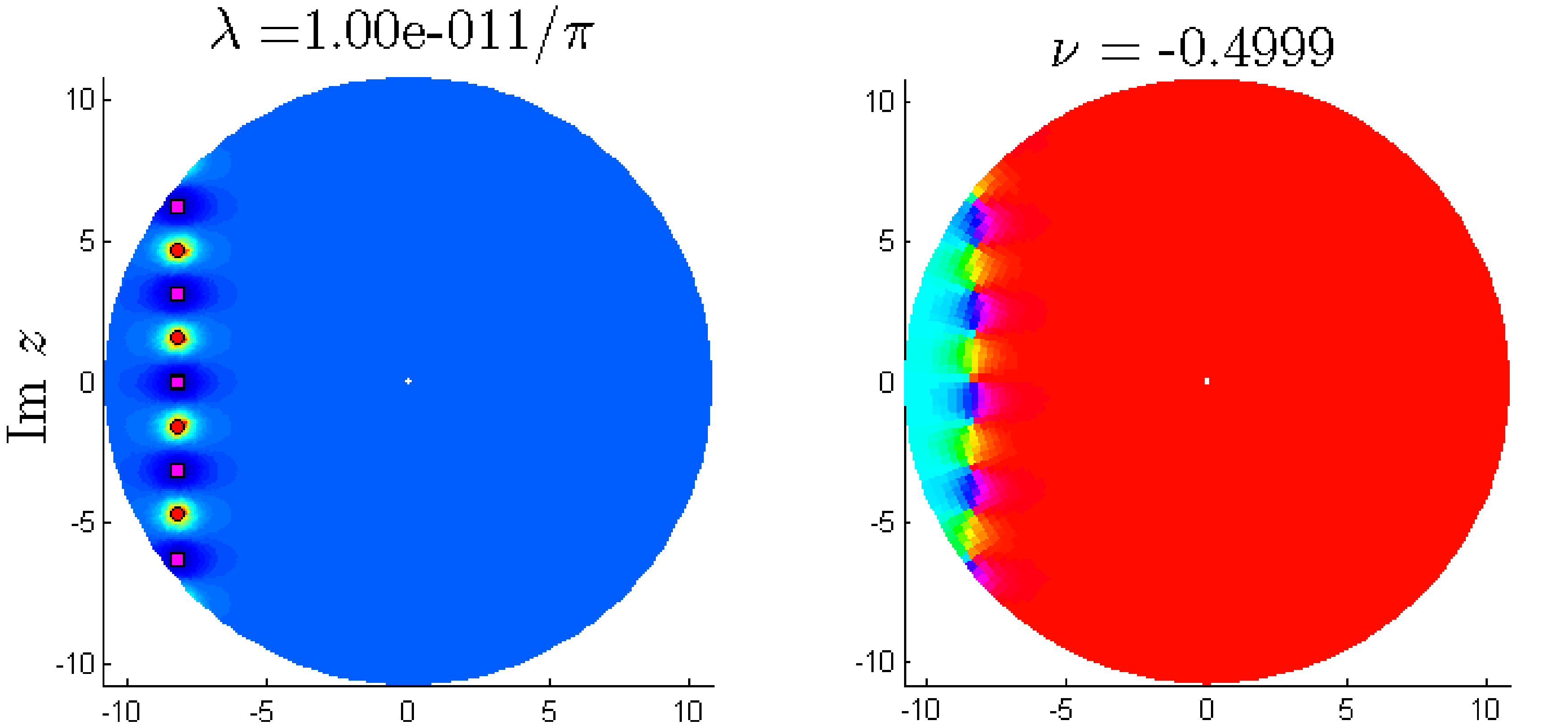

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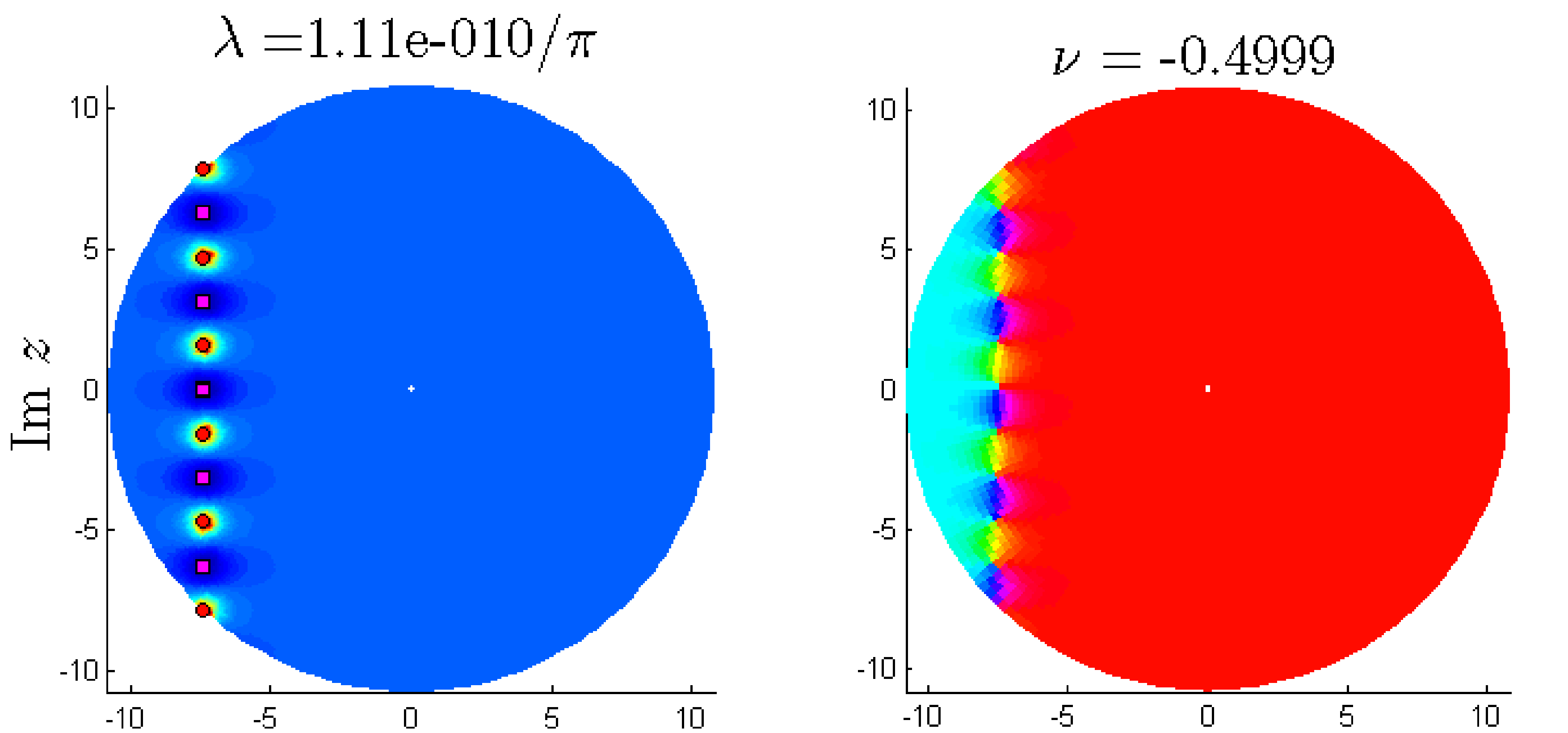

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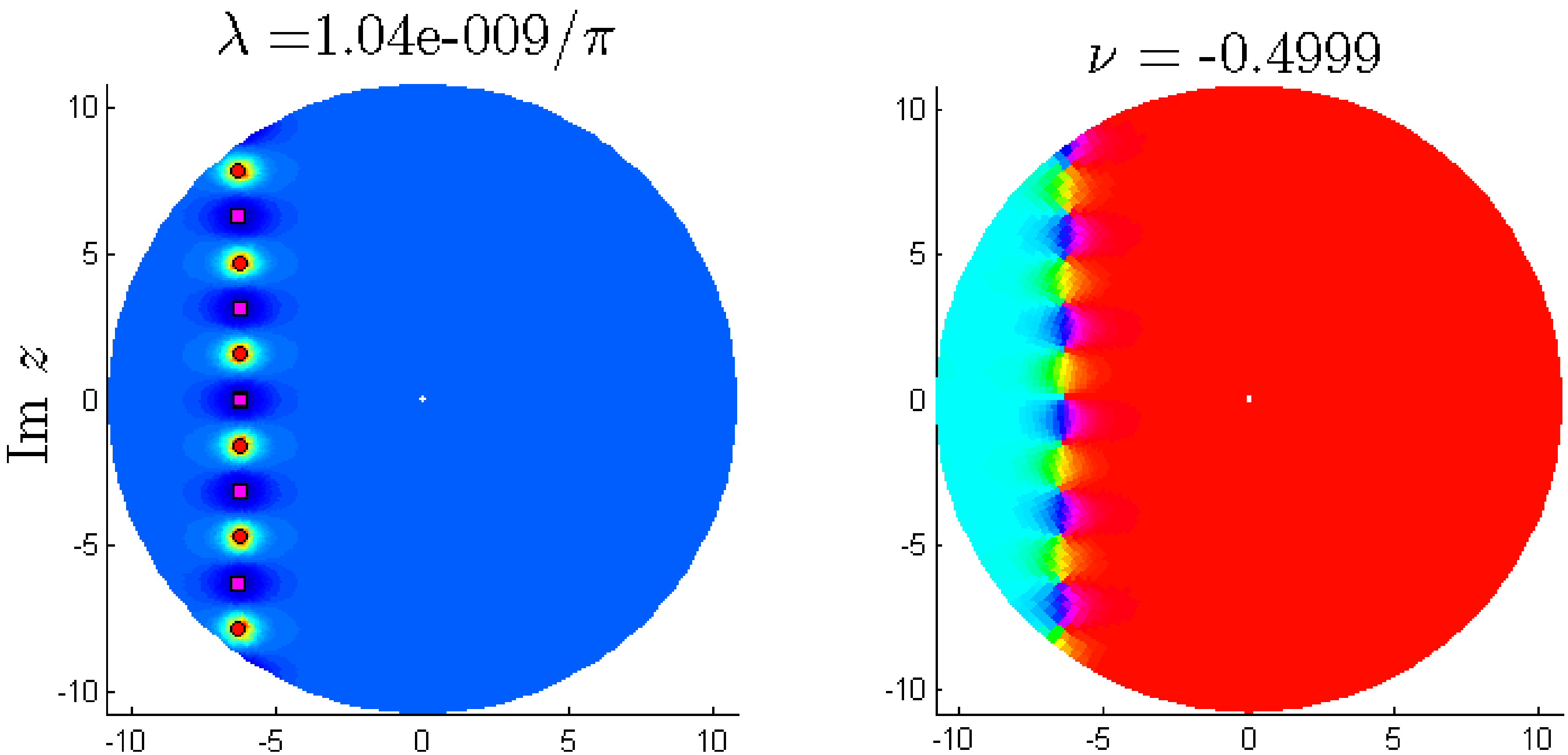

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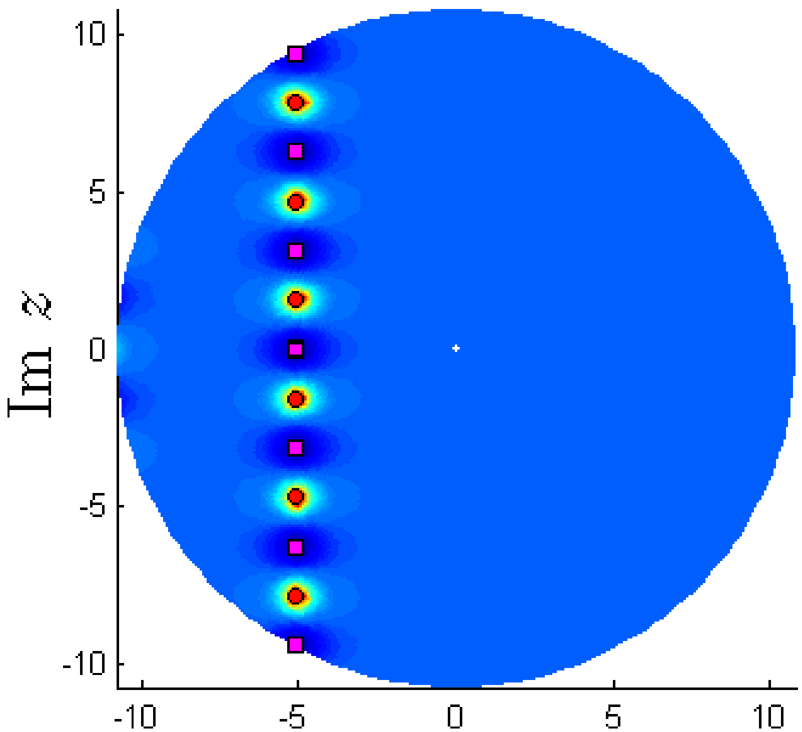
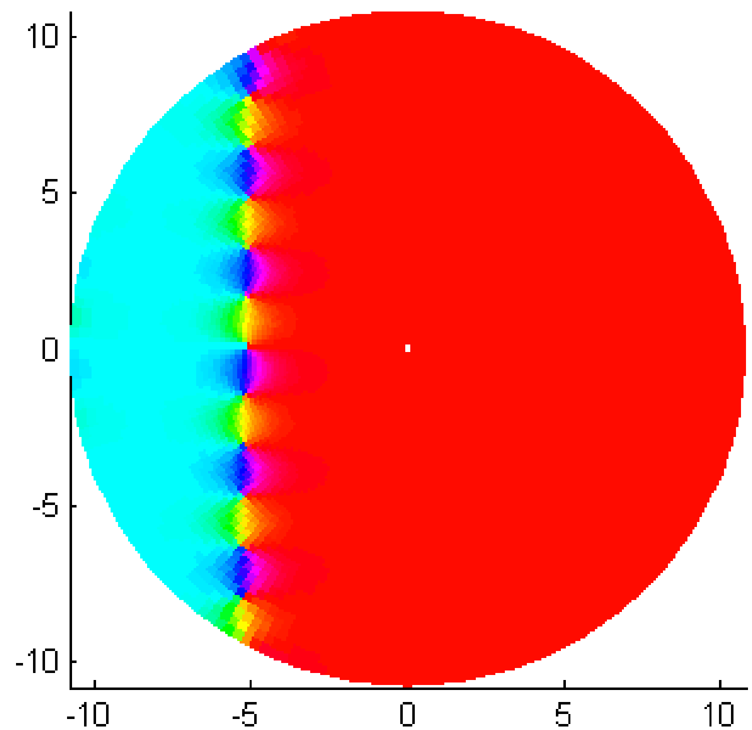
λ =1.16e-008/π



ν = -0.4999



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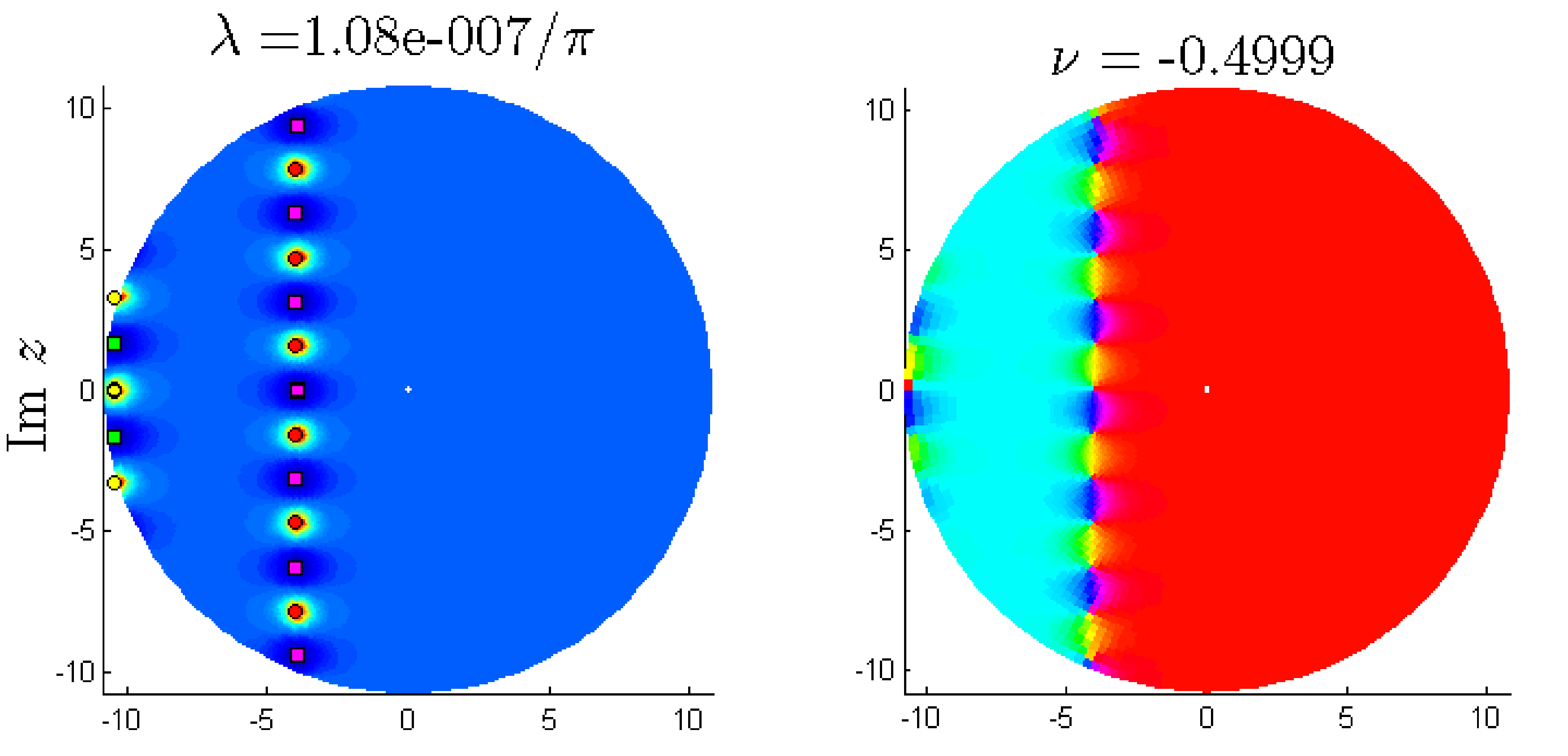

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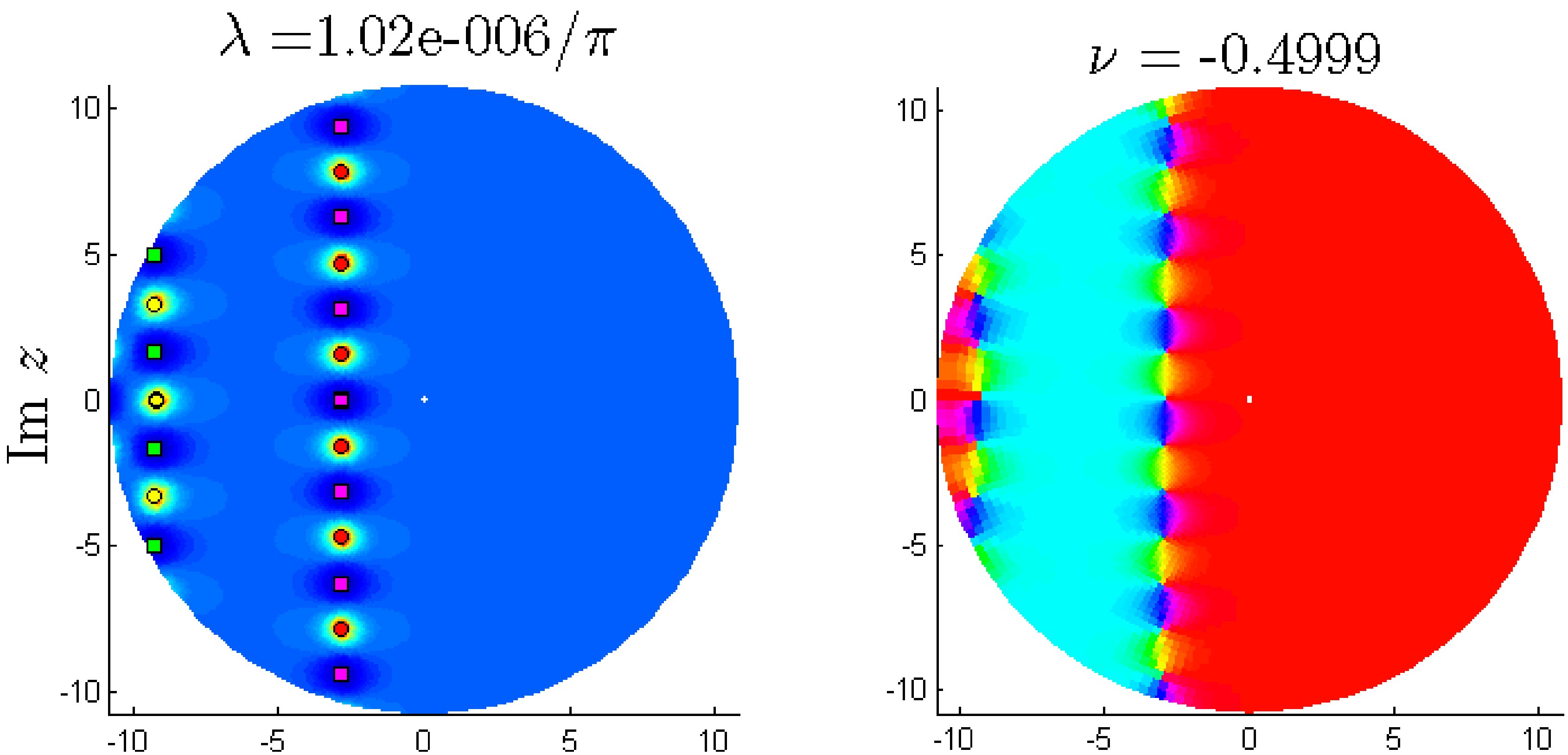

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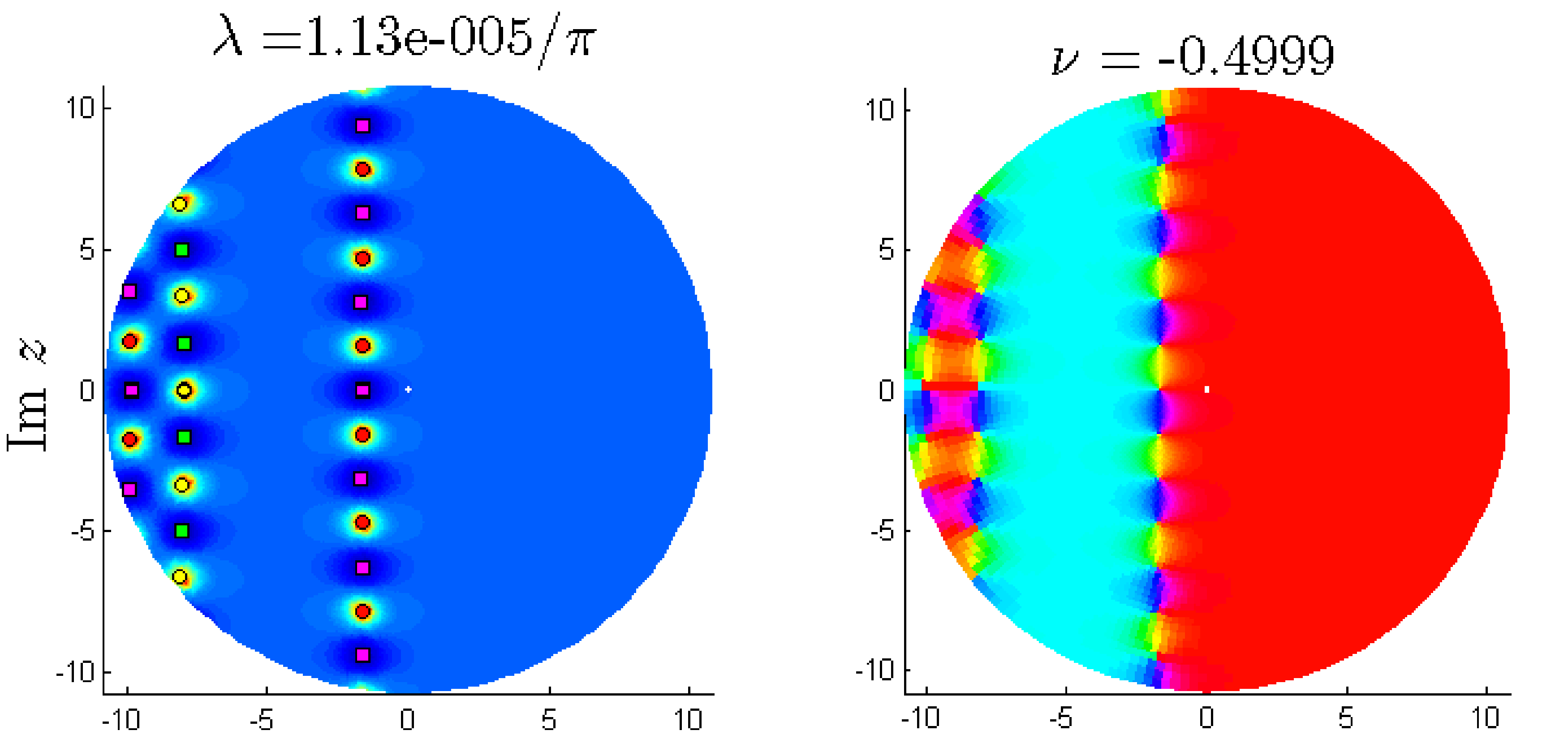

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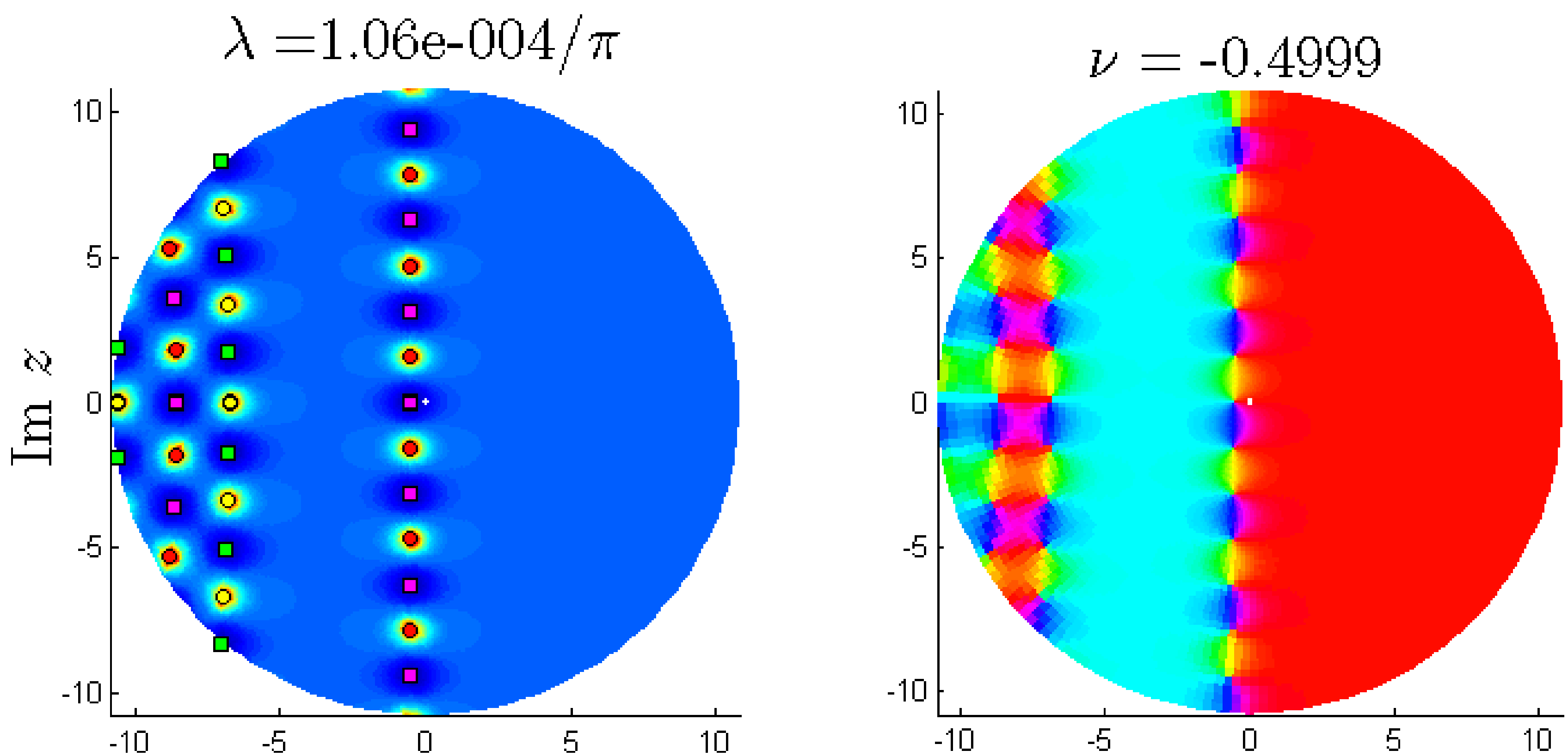

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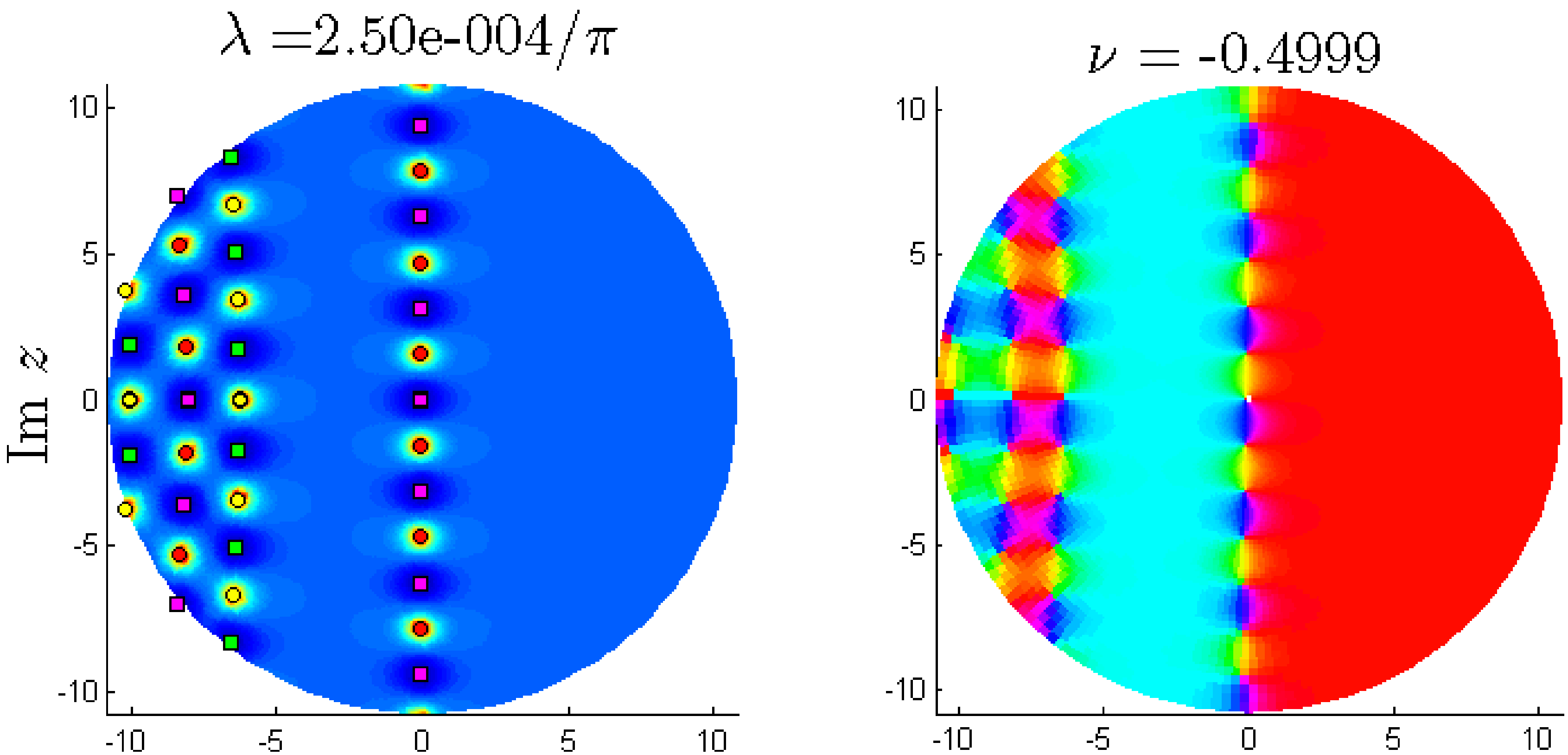

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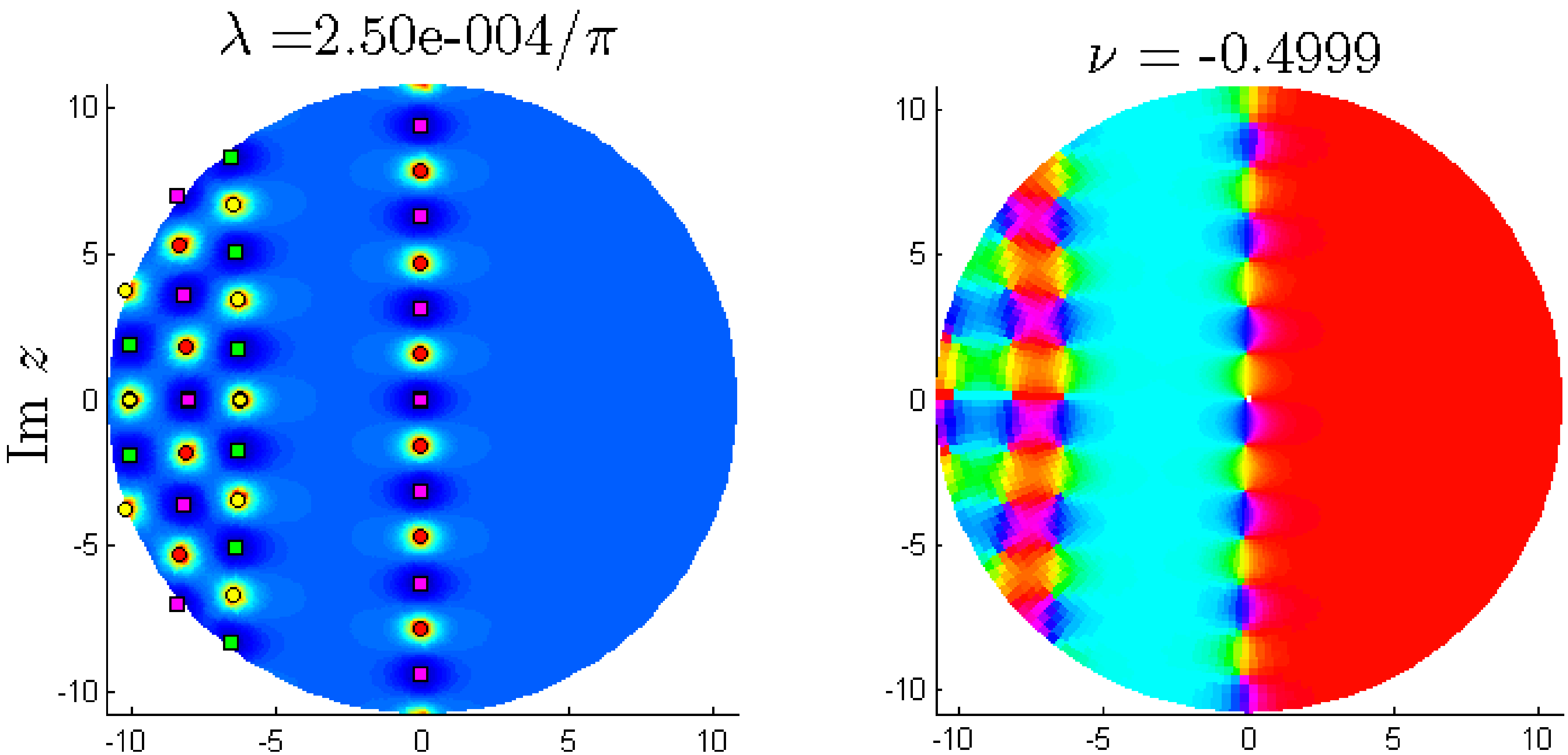

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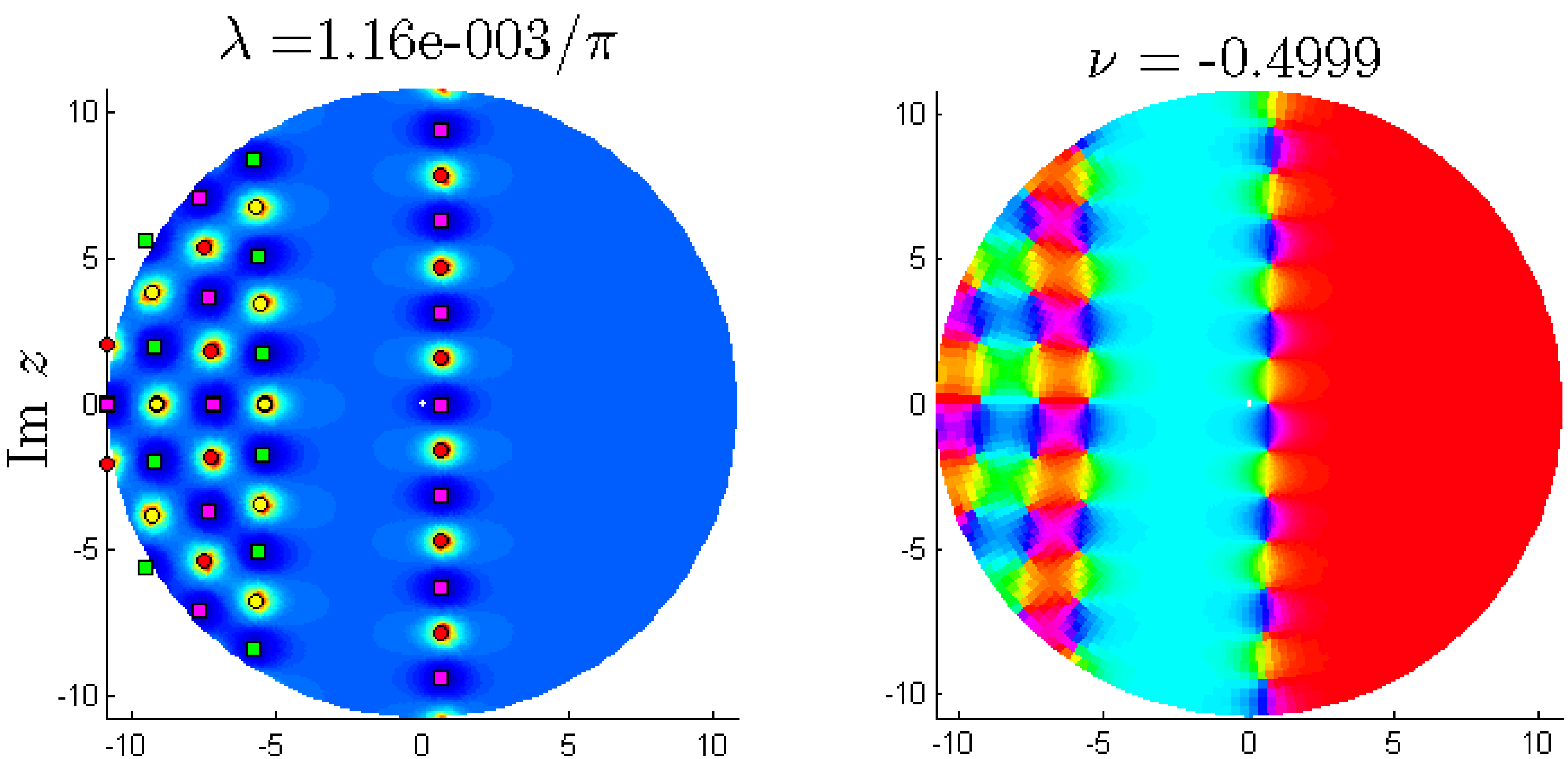

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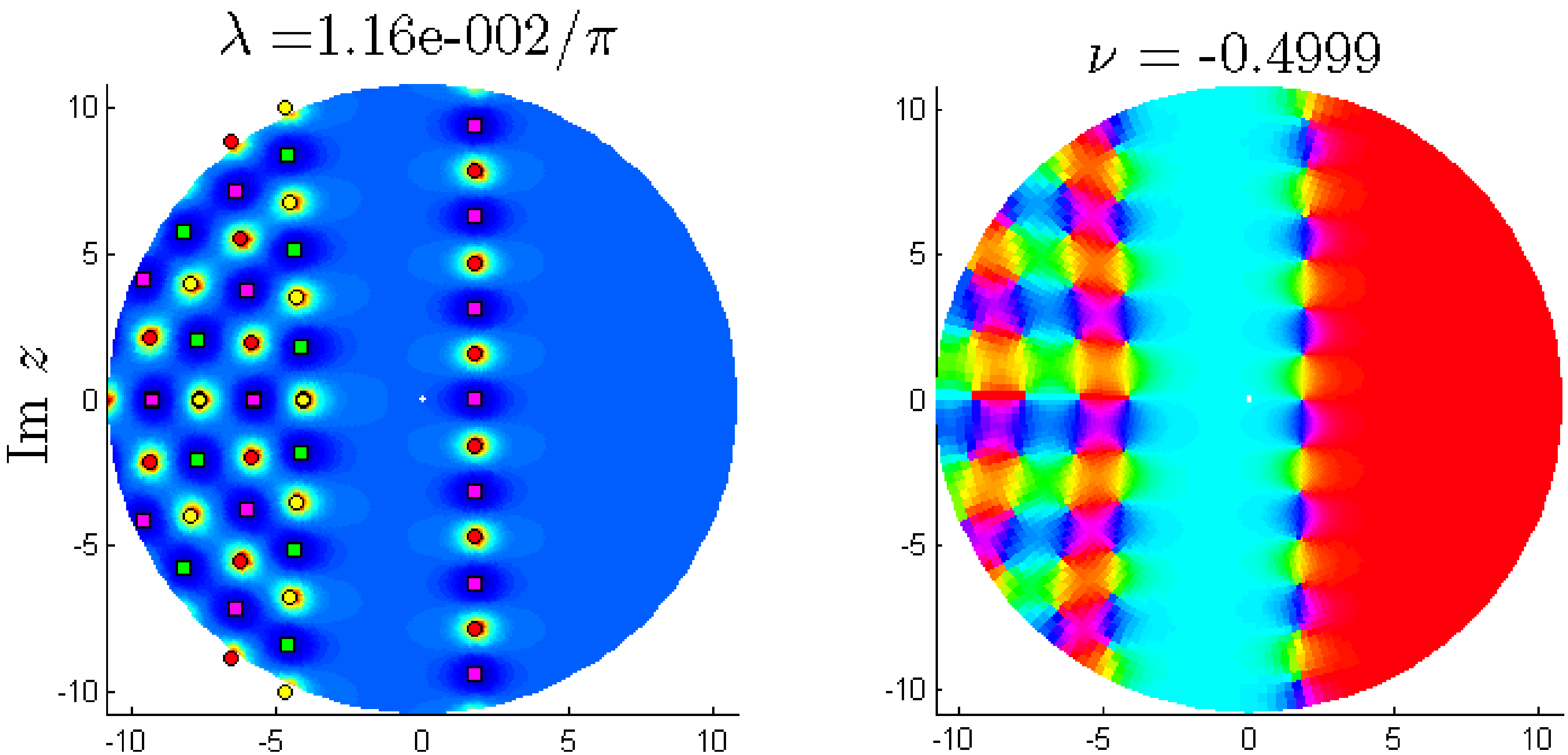

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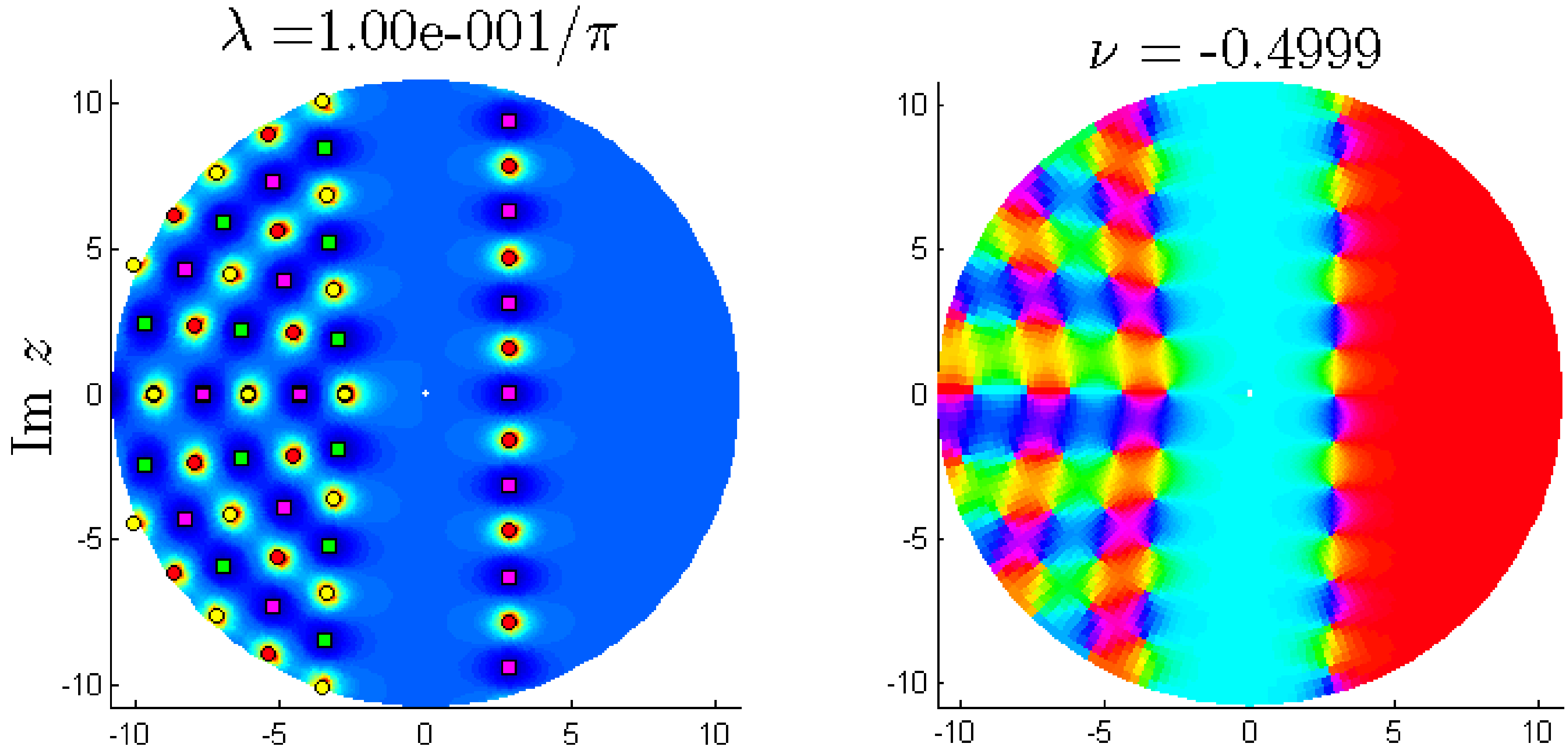

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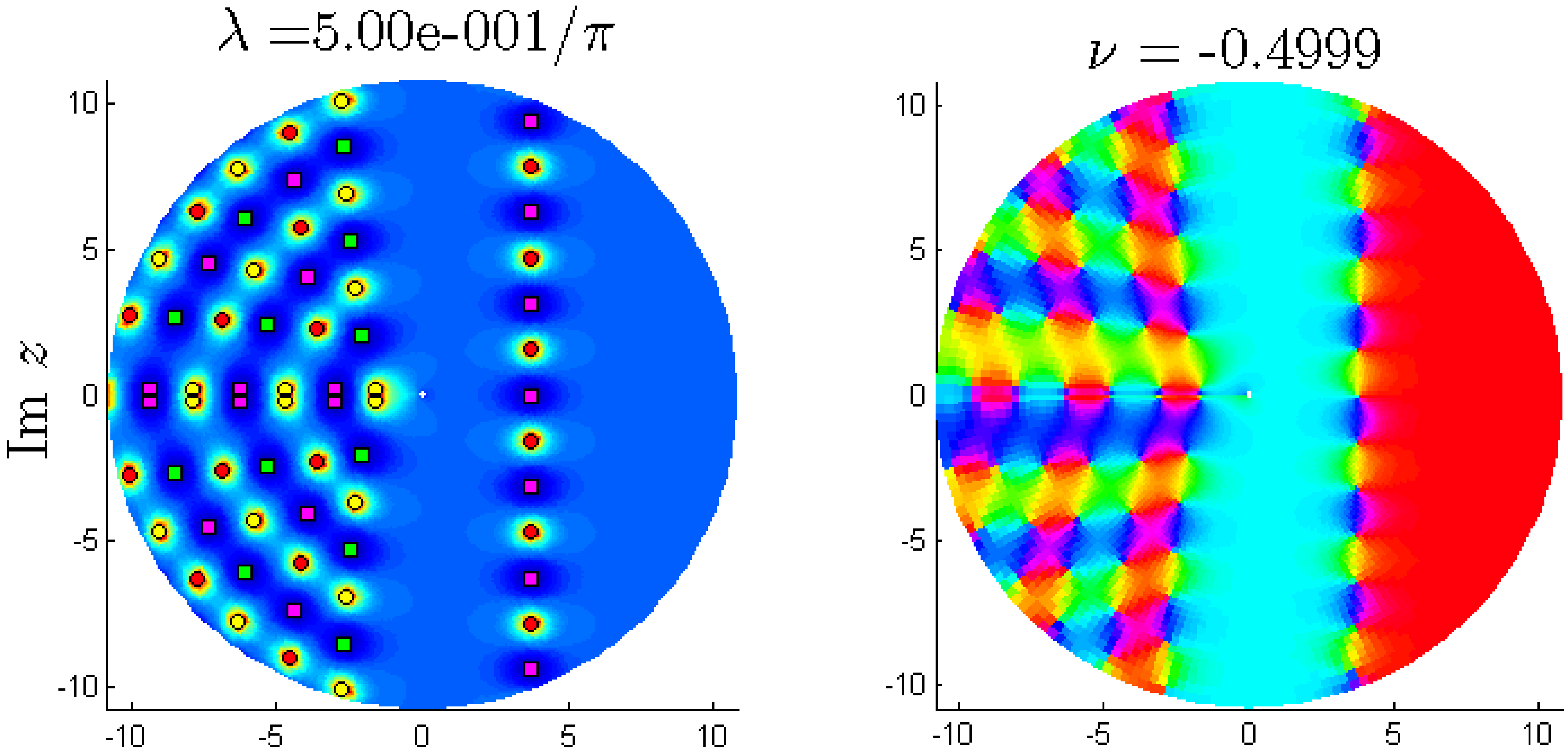

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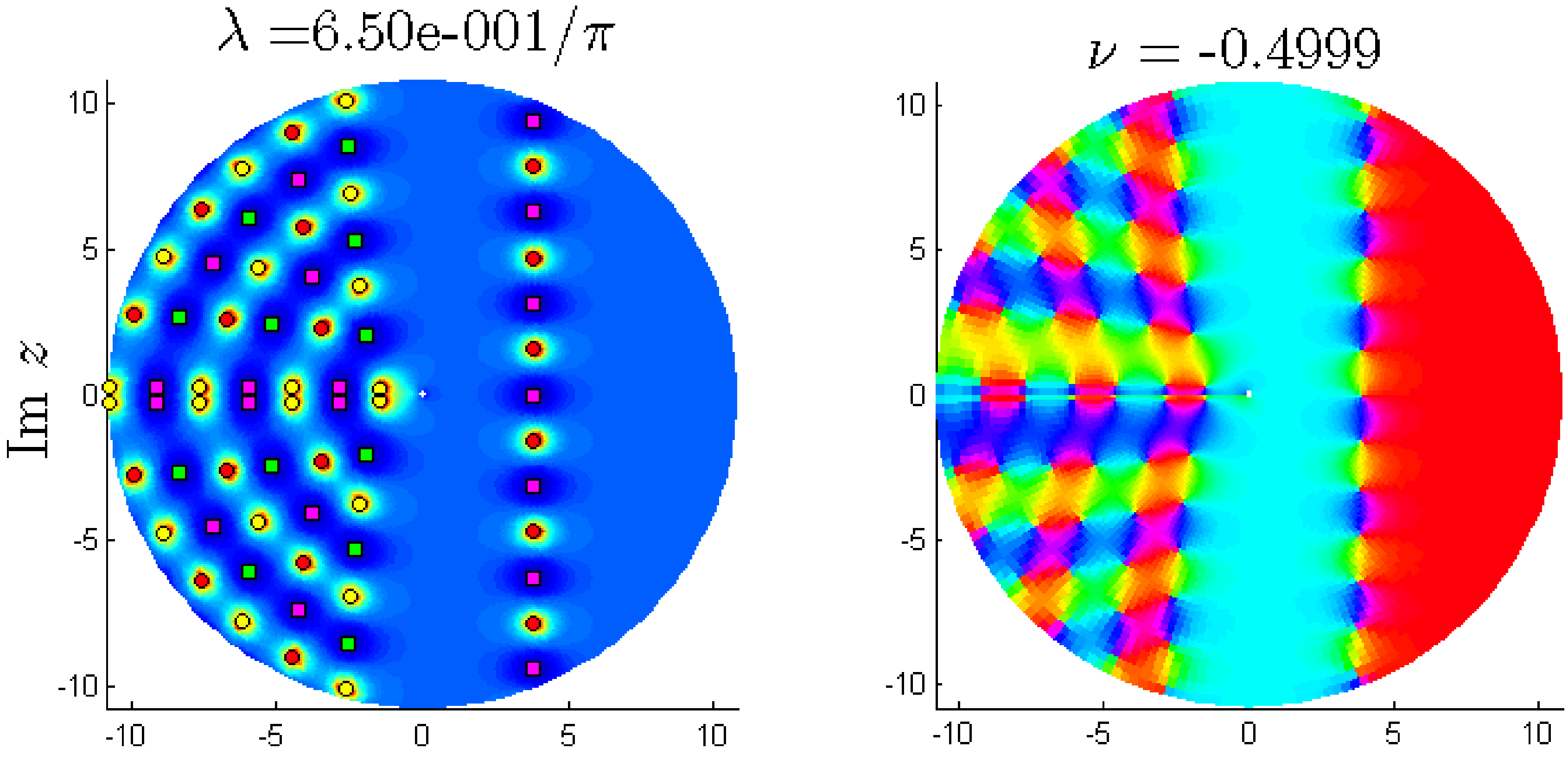

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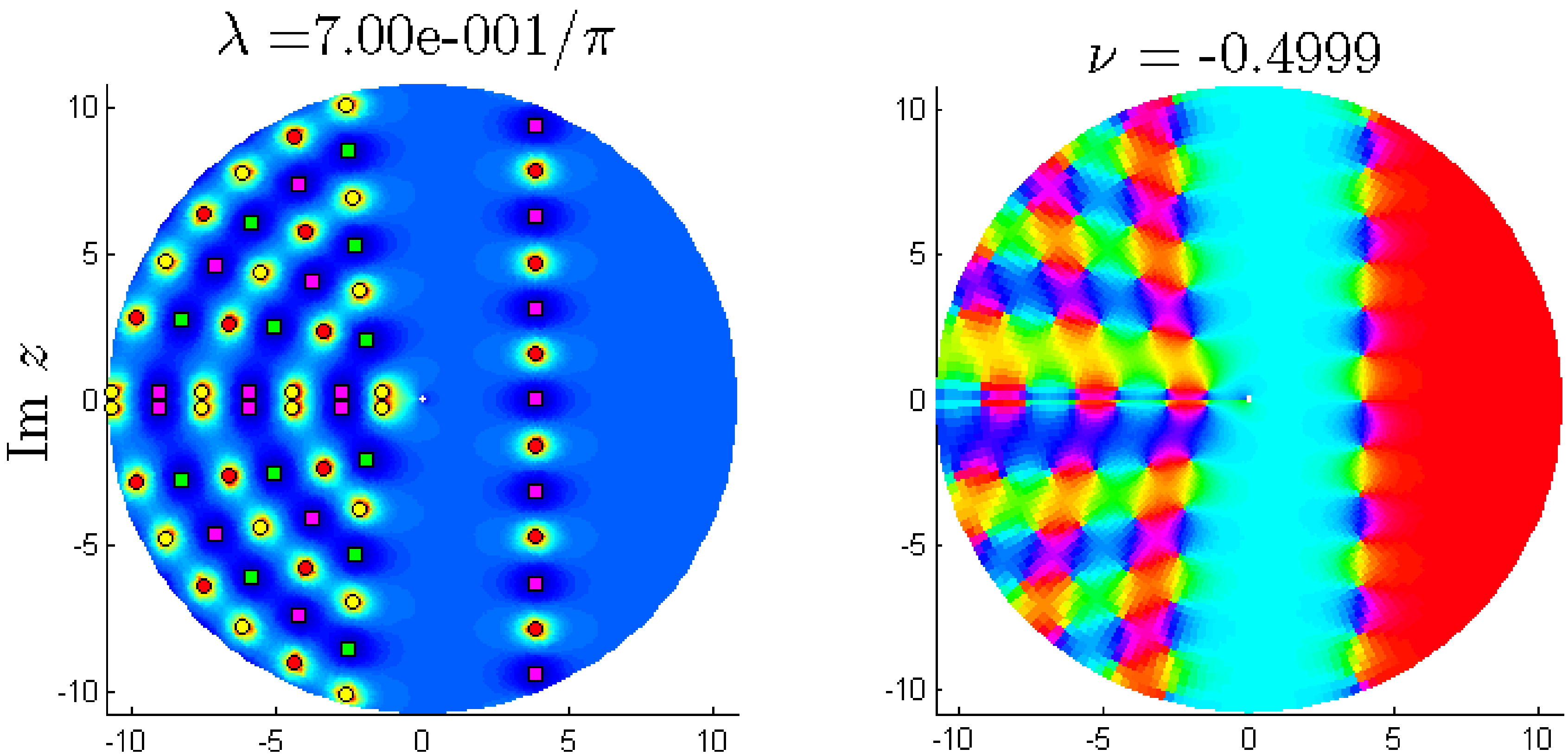

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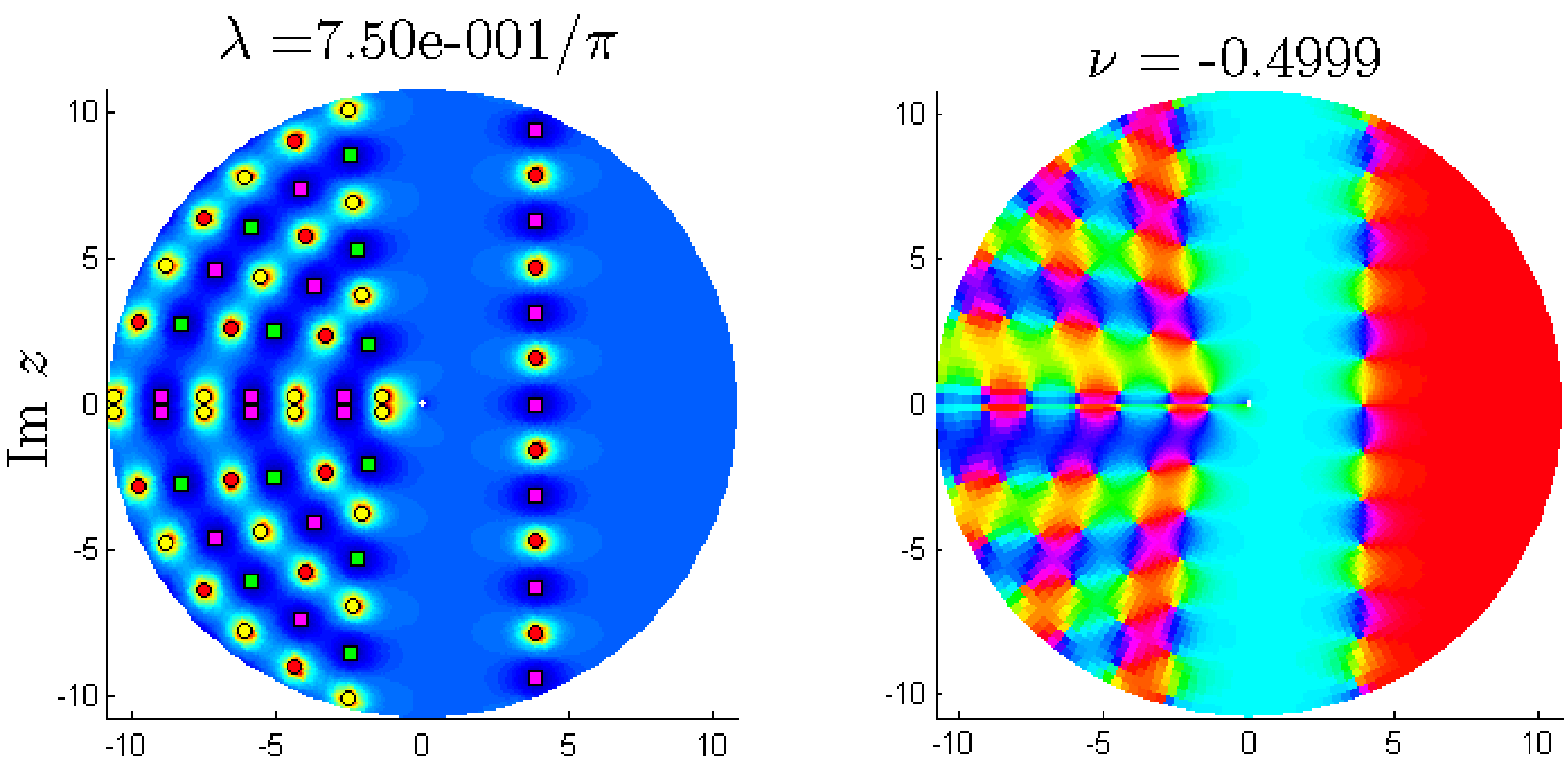

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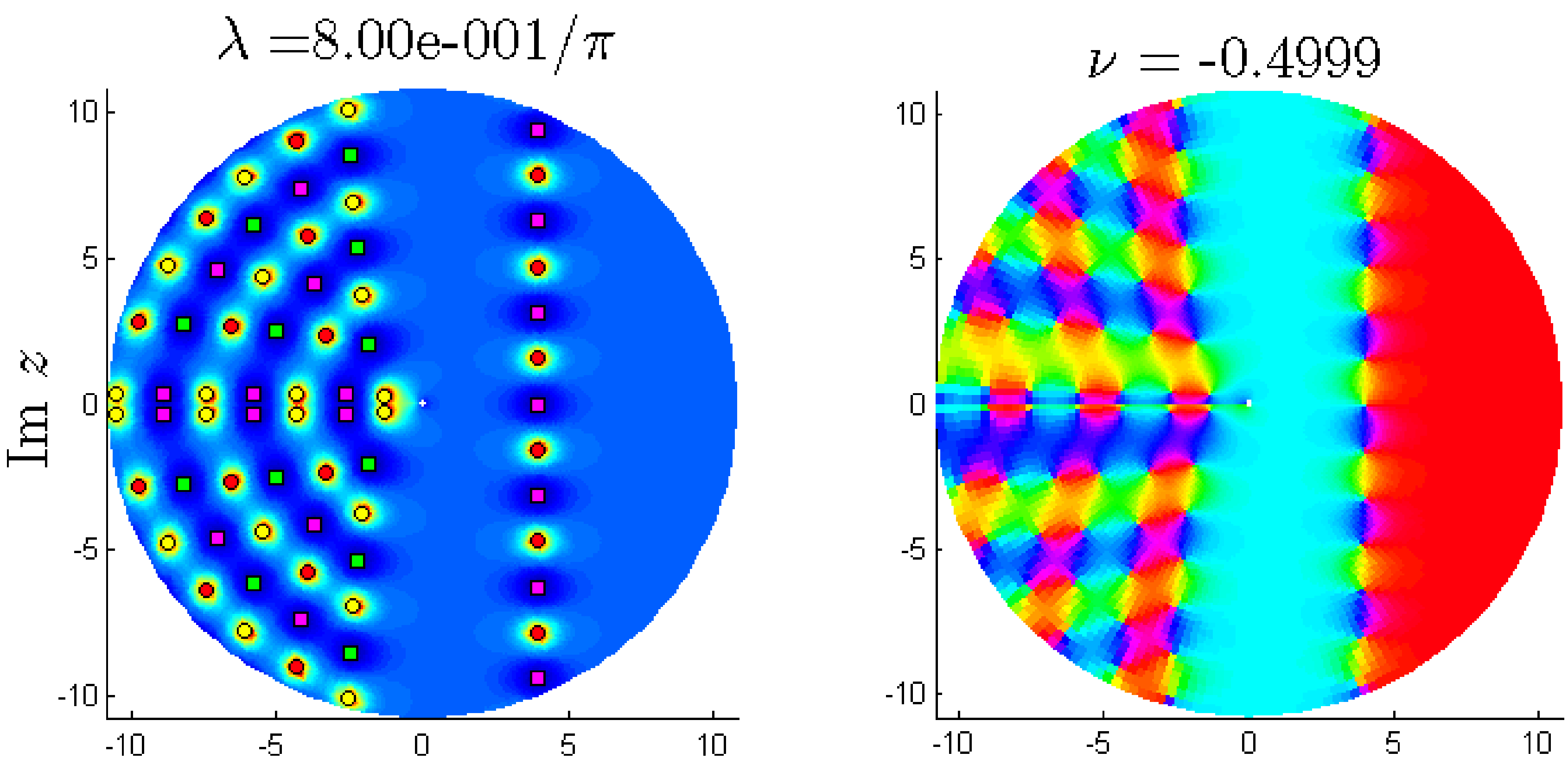

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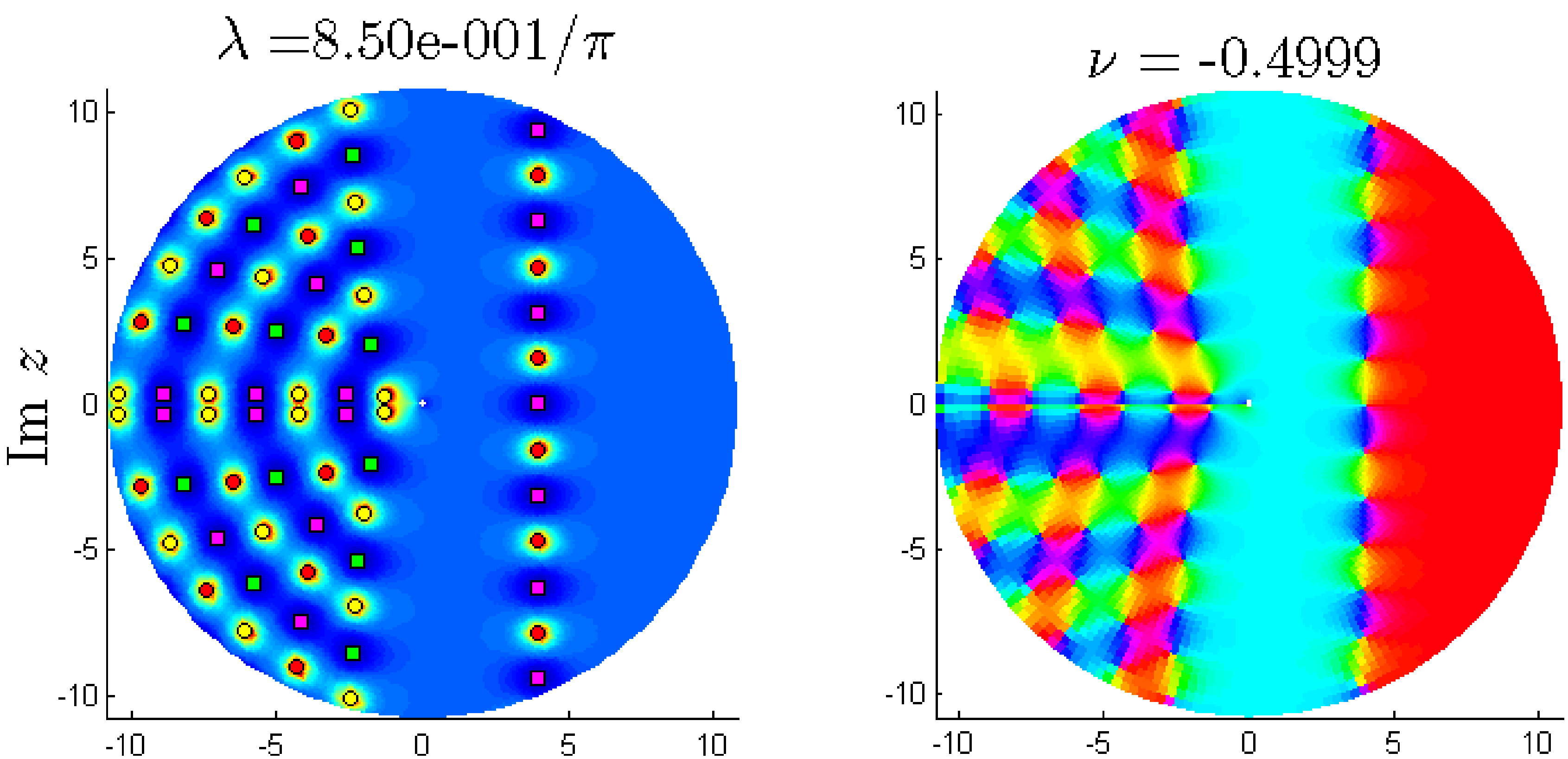
The MTW solutions: $\boldsymbol{\alpha} = -\boldsymbol{\beta} = 2\boldsymbol{\nu}$, $\boldsymbol{\gamma} = 1$, $\boldsymbol{\delta} = -1$

$$u(x) \sim 1 - \lambda\,\Gamma\left(\nu + \tfrac{1}{2}\right) 2^{-2\nu} x^{-\nu-(1/2)} e^{-2x}, \qquad x \to \infty$$

$$u(x) \sim B x^{\sigma}, \qquad x \to 0^+, \qquad -\frac{1}{\pi} < \lambda < \frac{1}{\pi}$$



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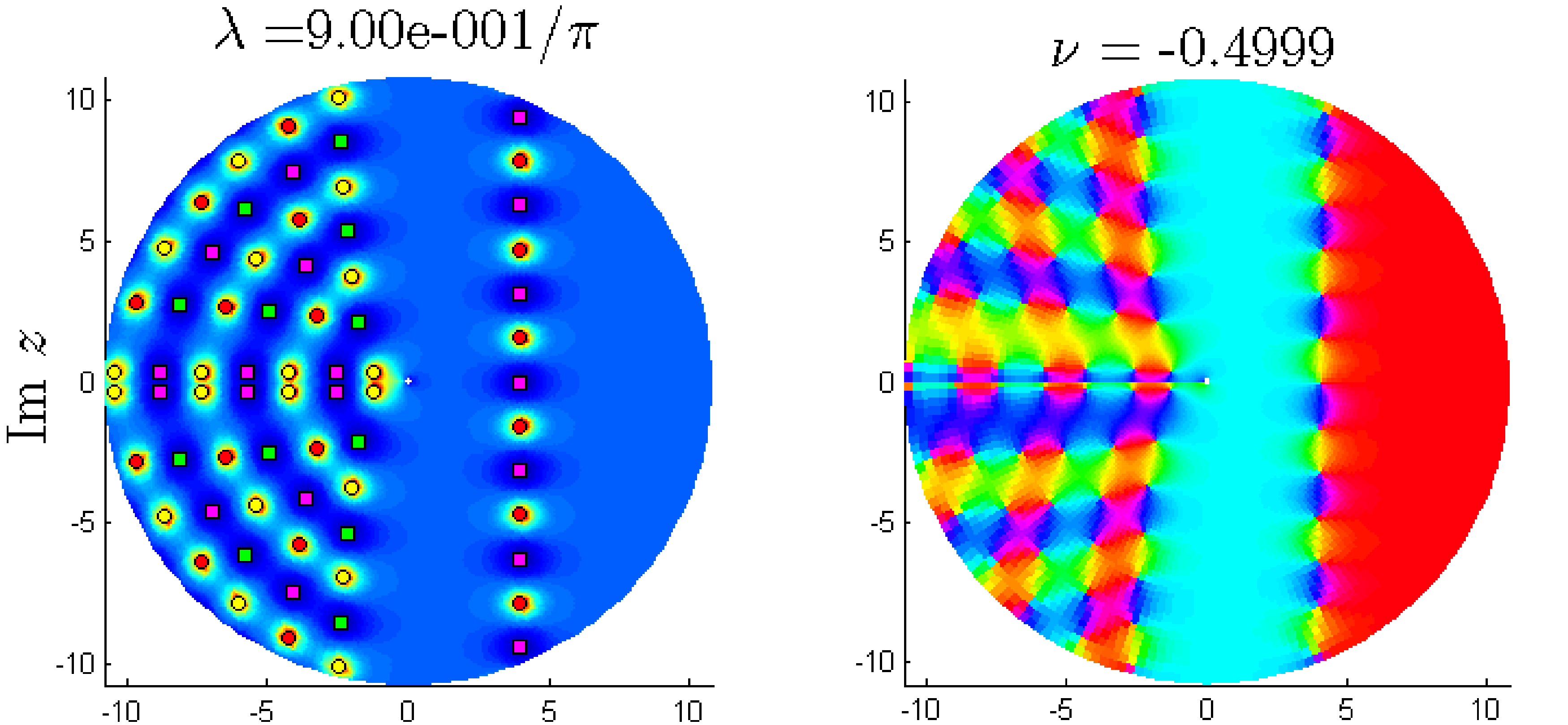

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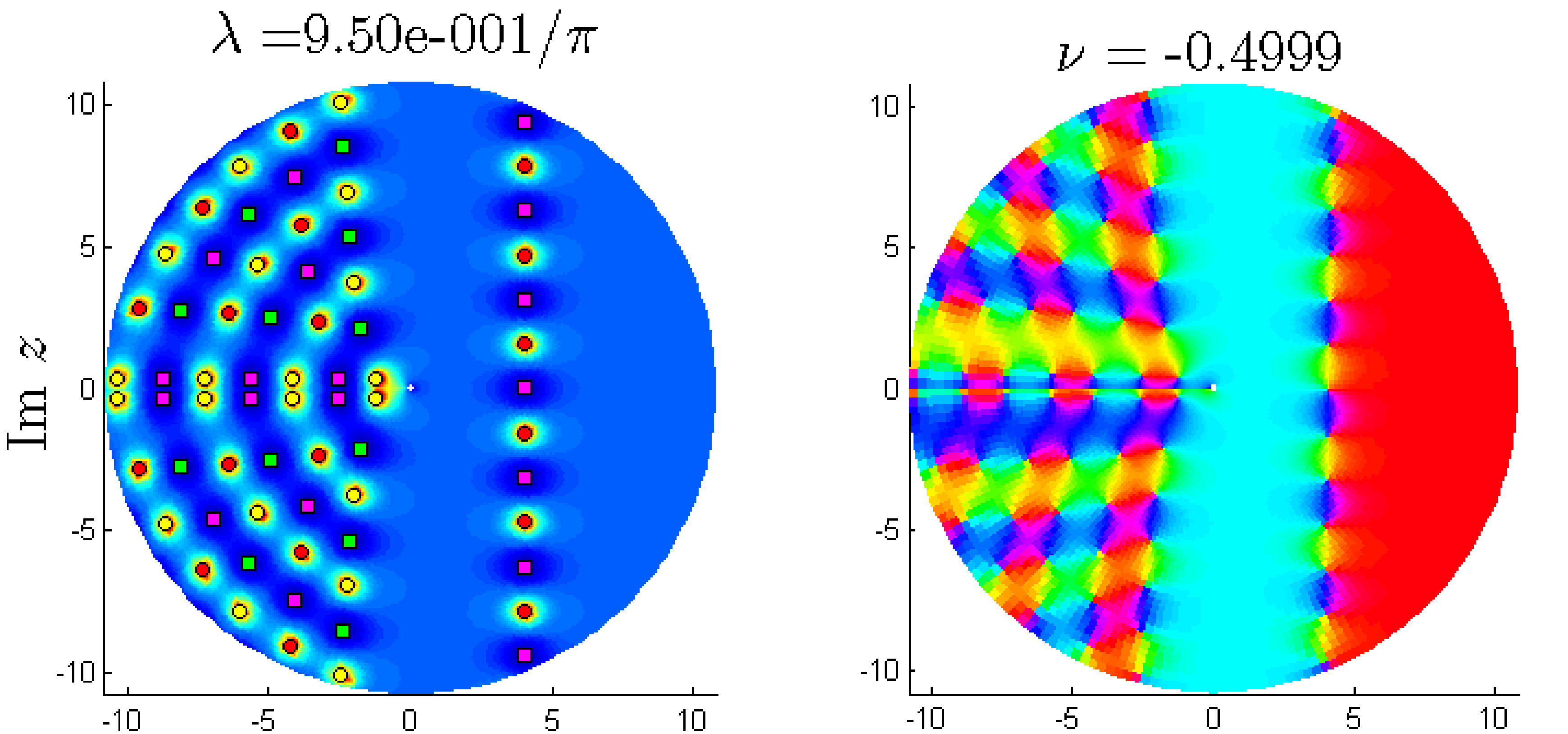

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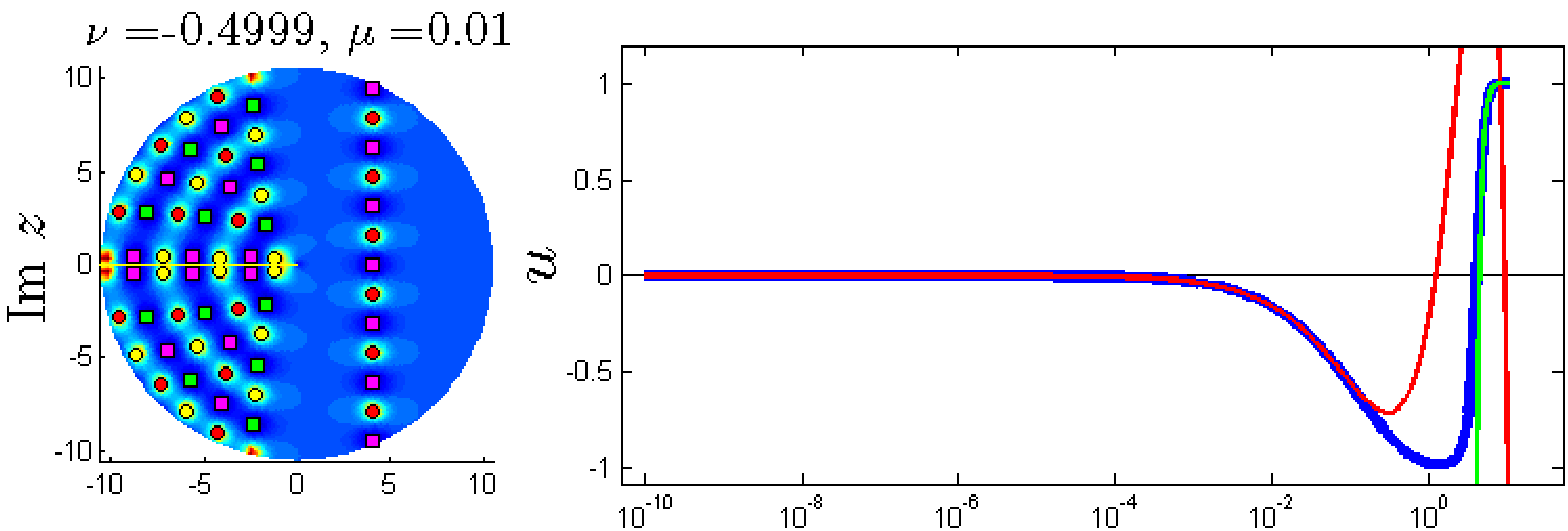

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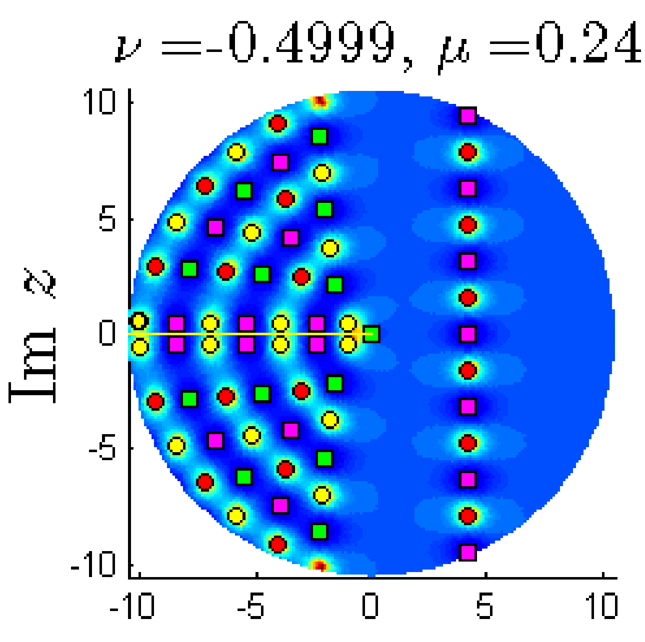

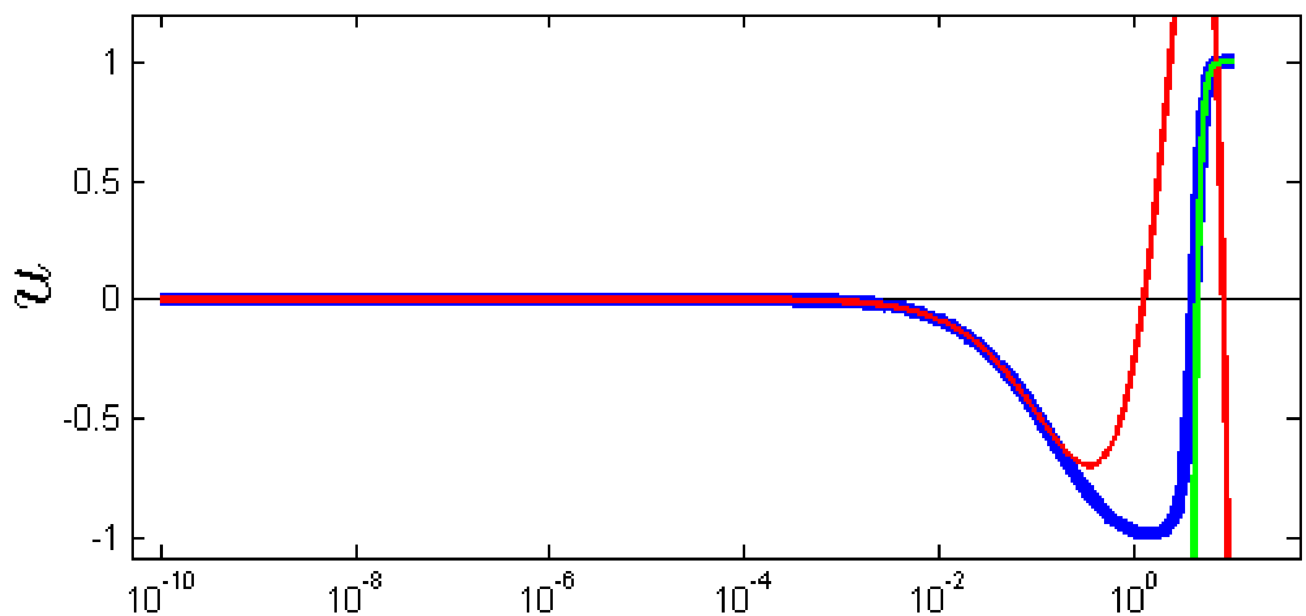

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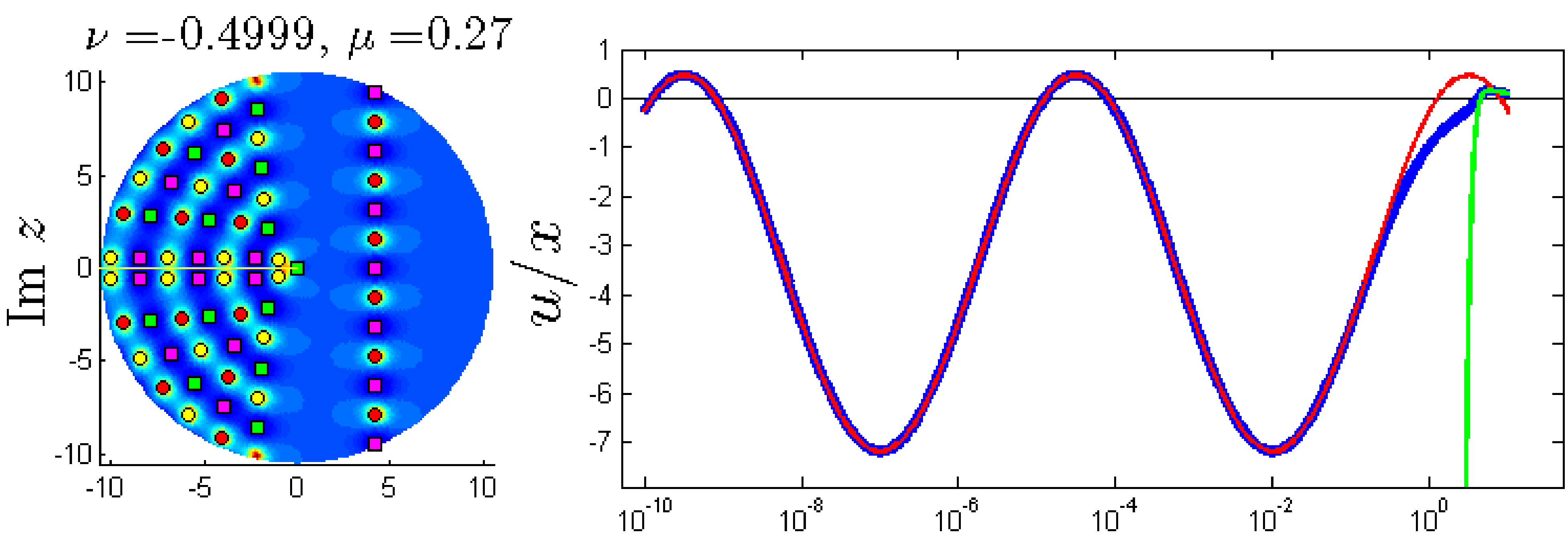

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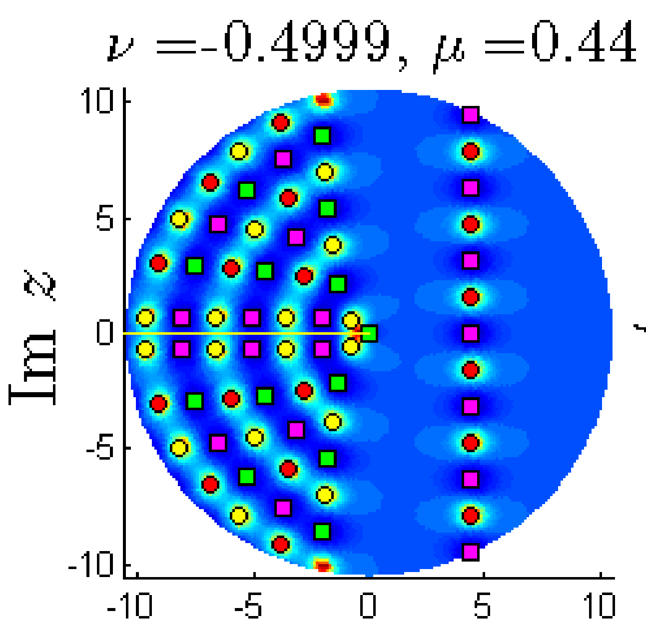

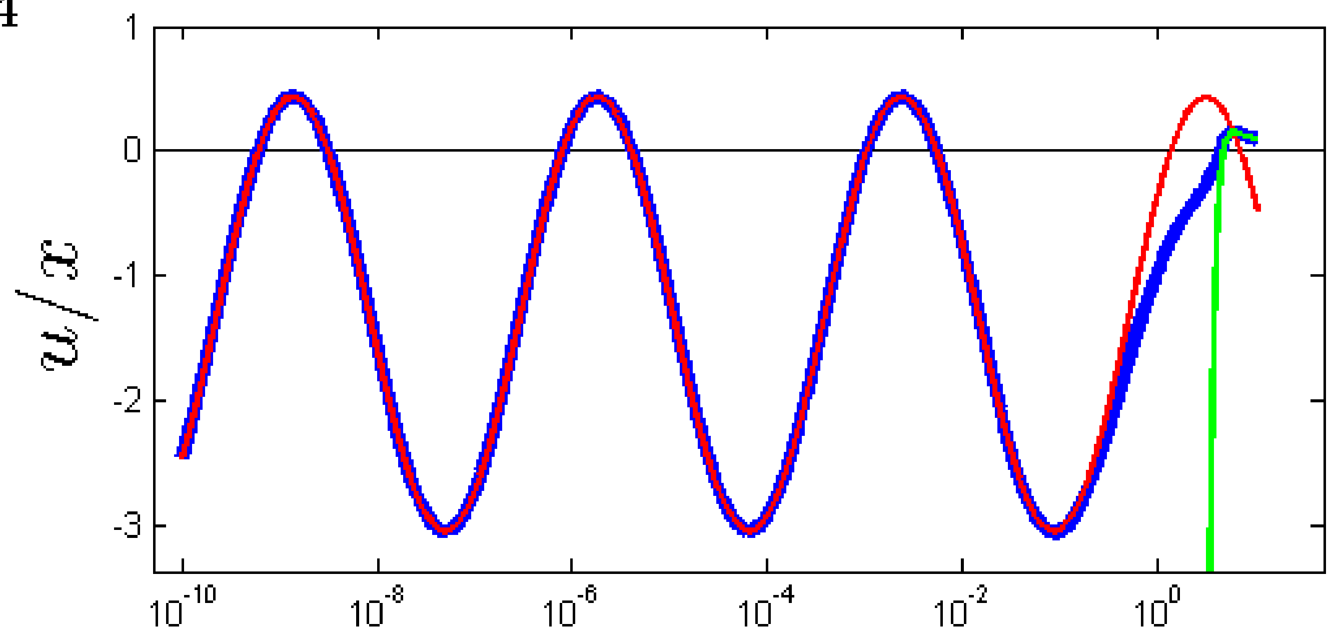

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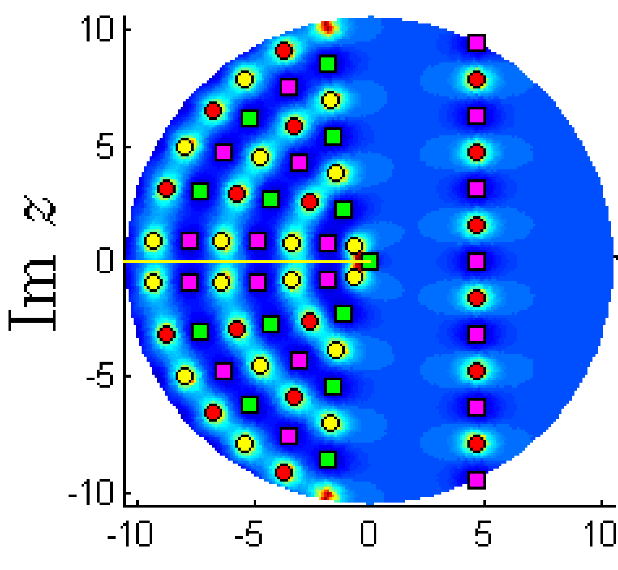

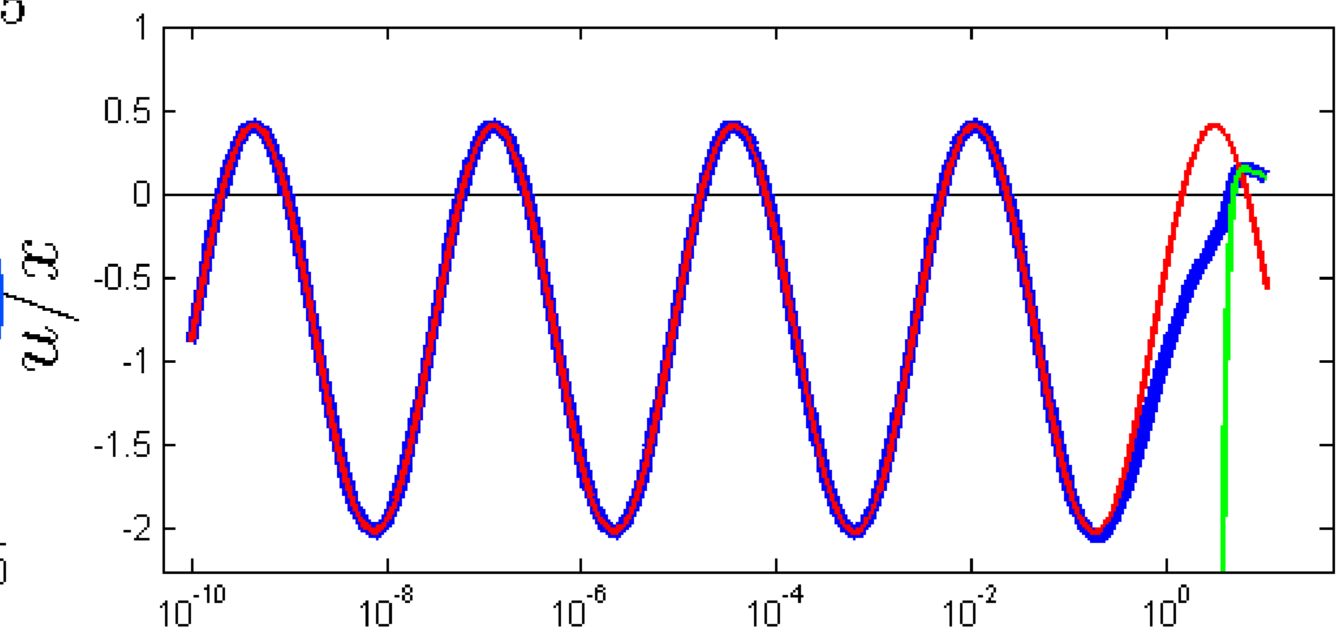

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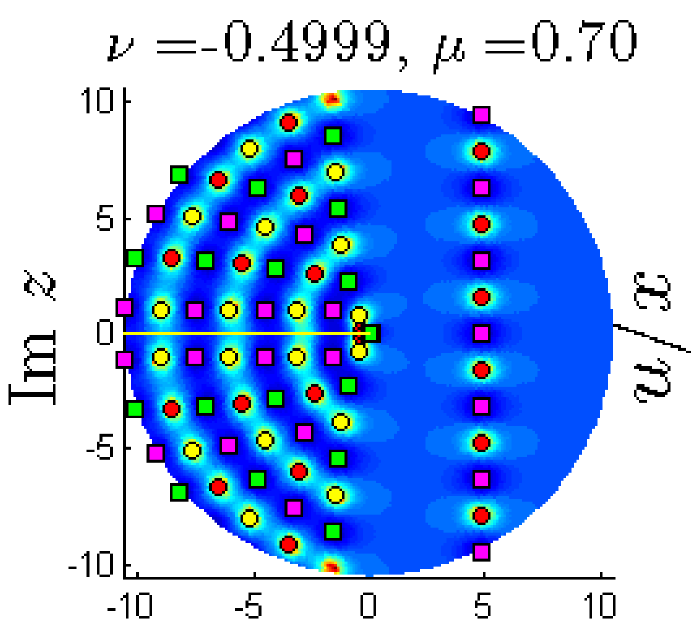

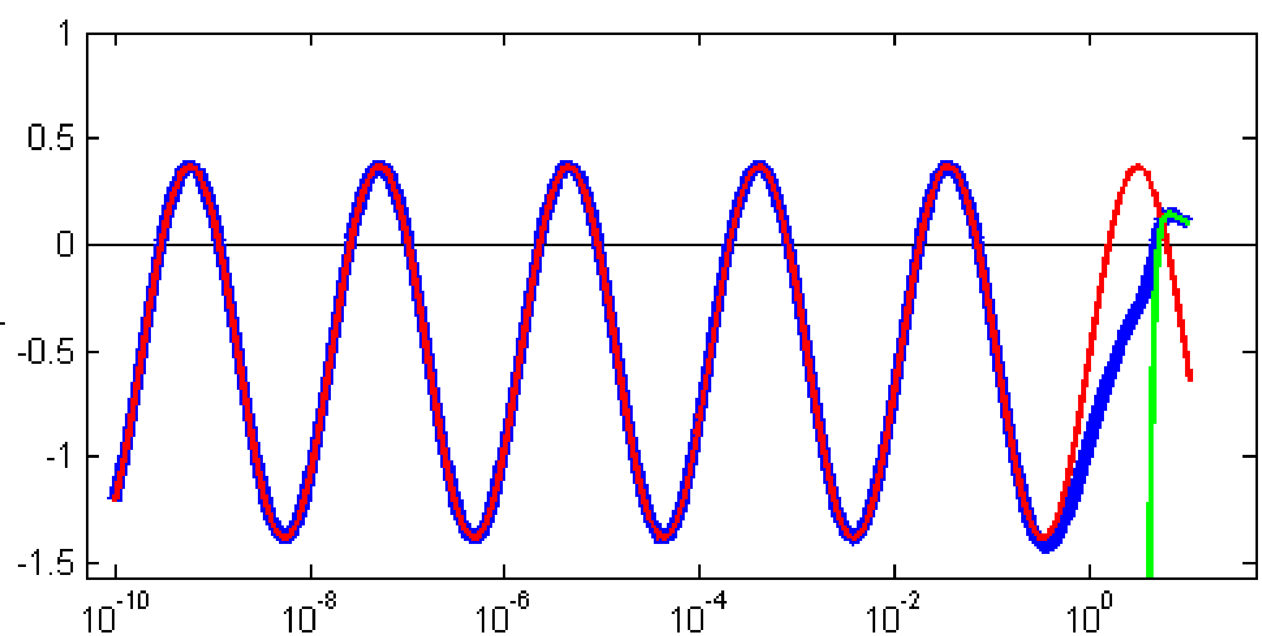

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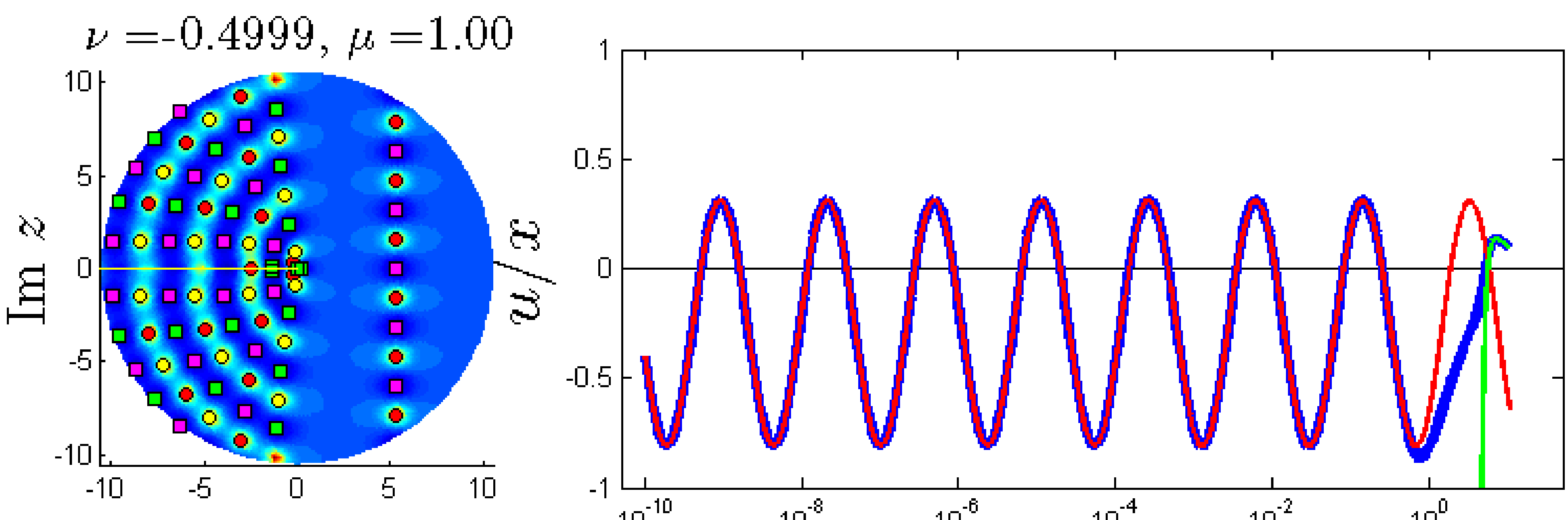

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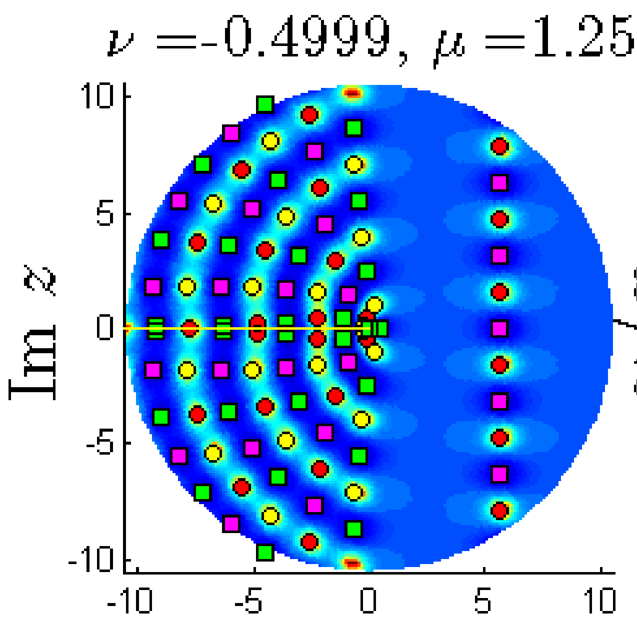

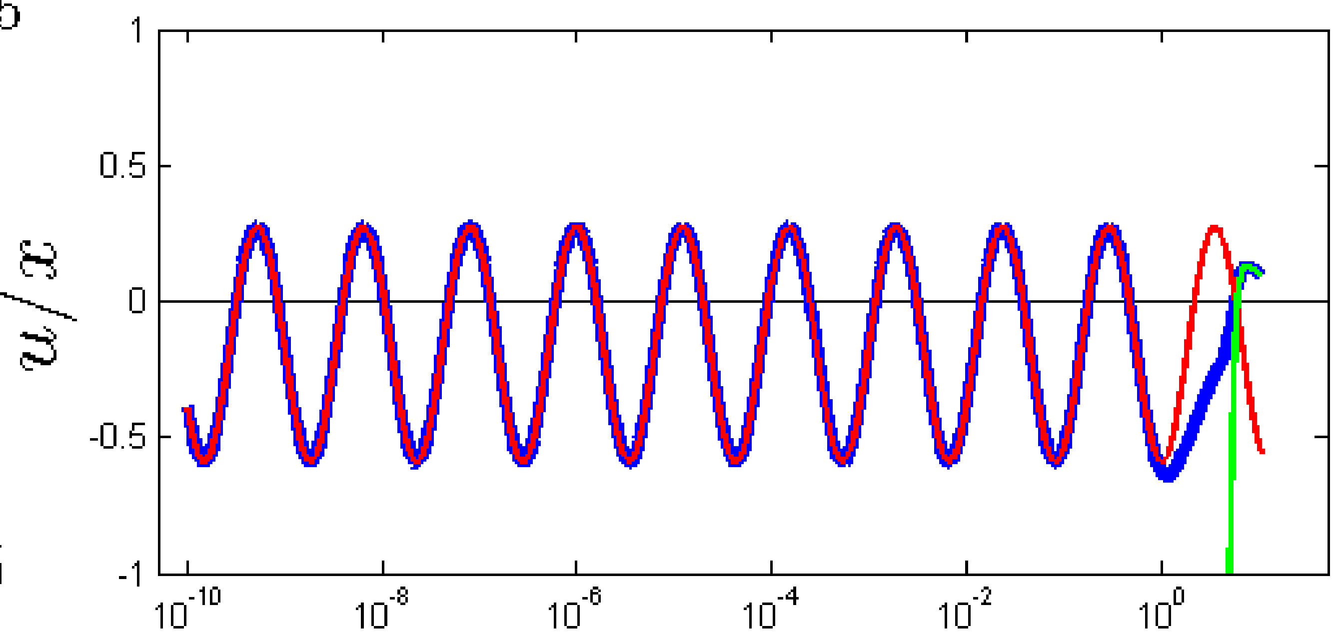

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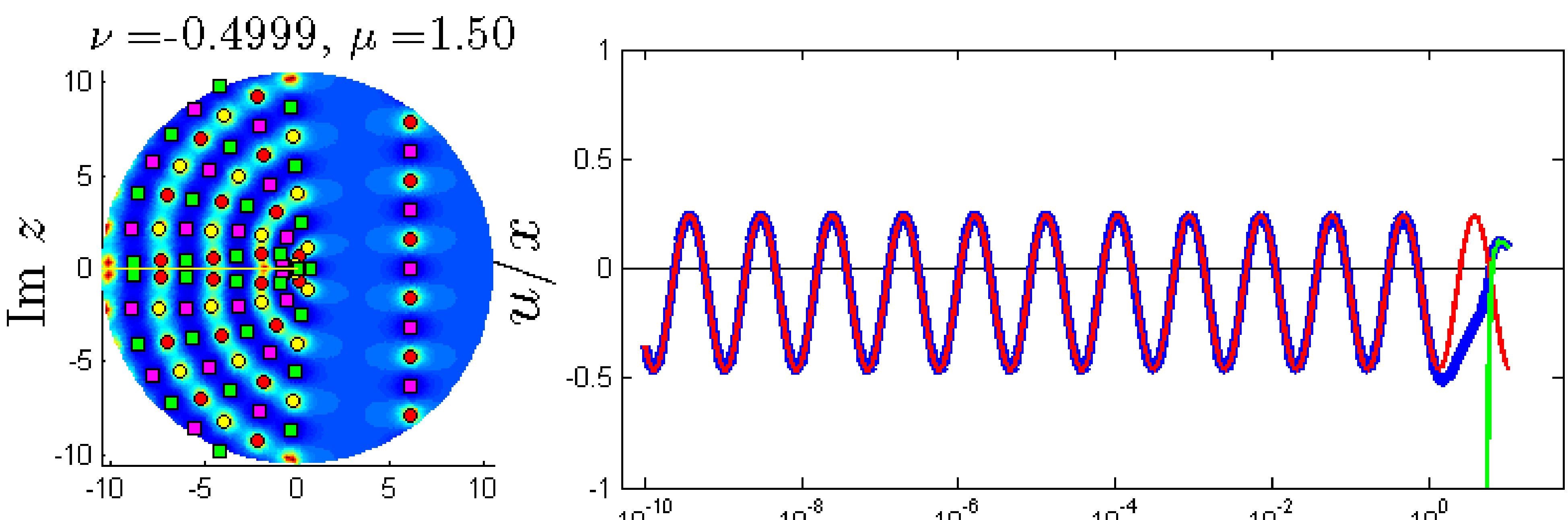

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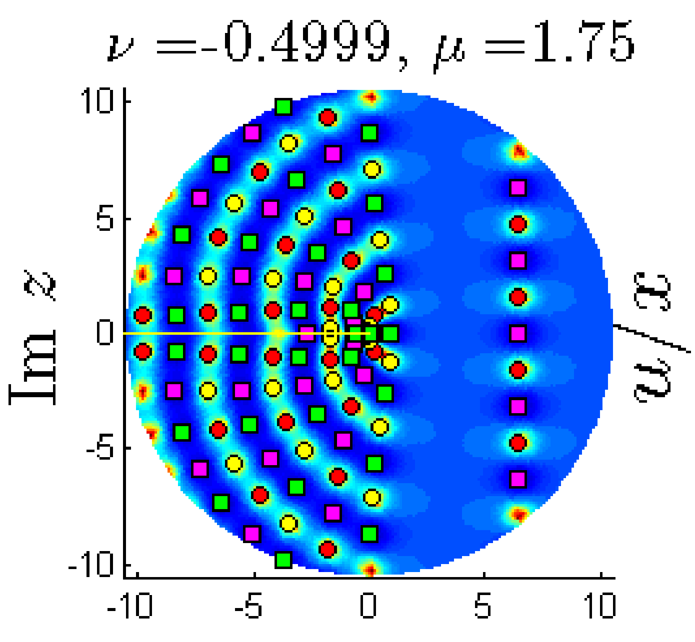

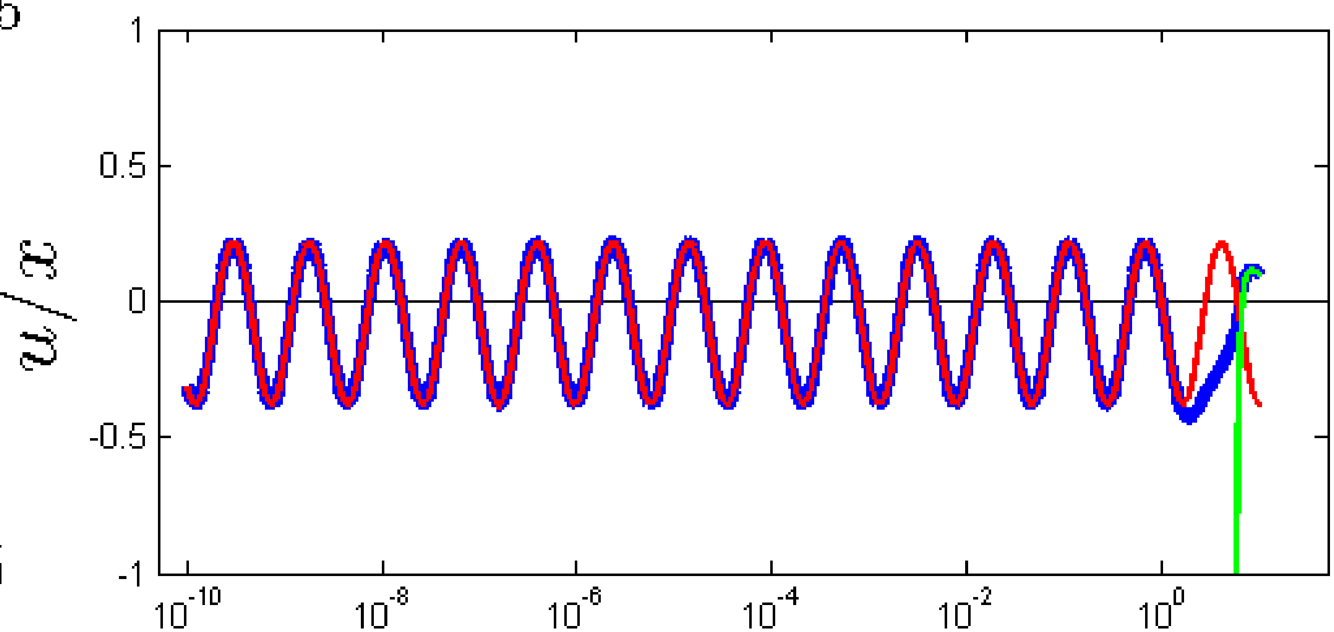

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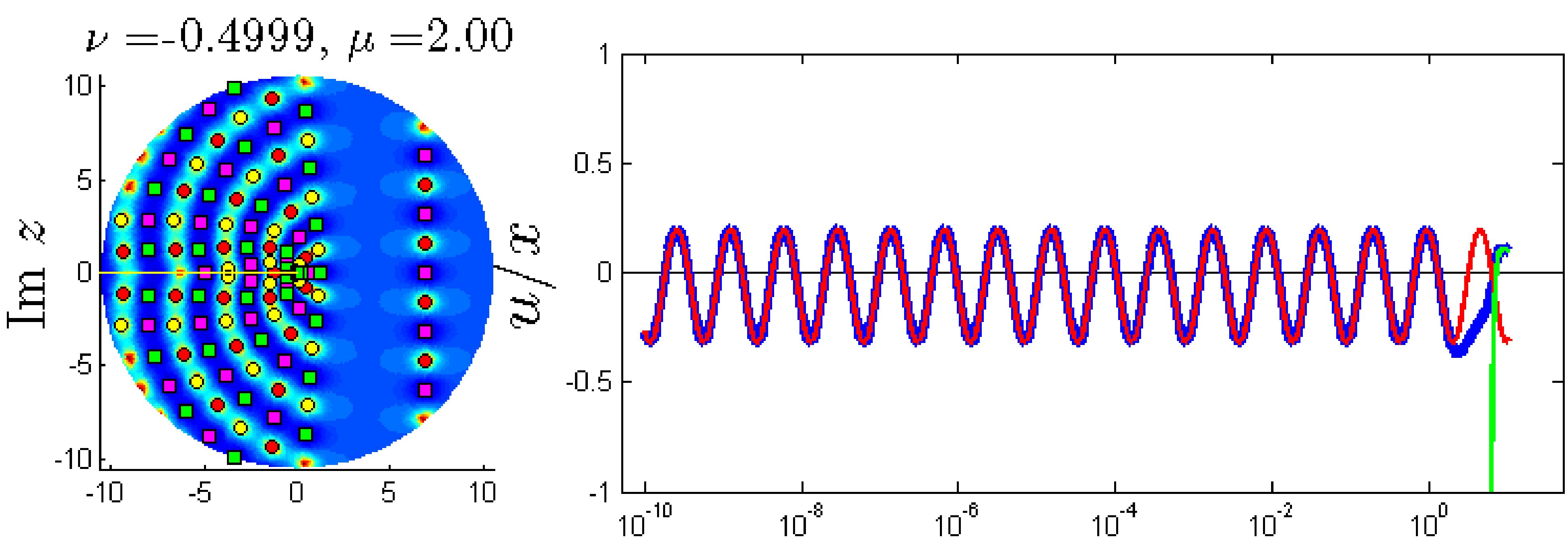

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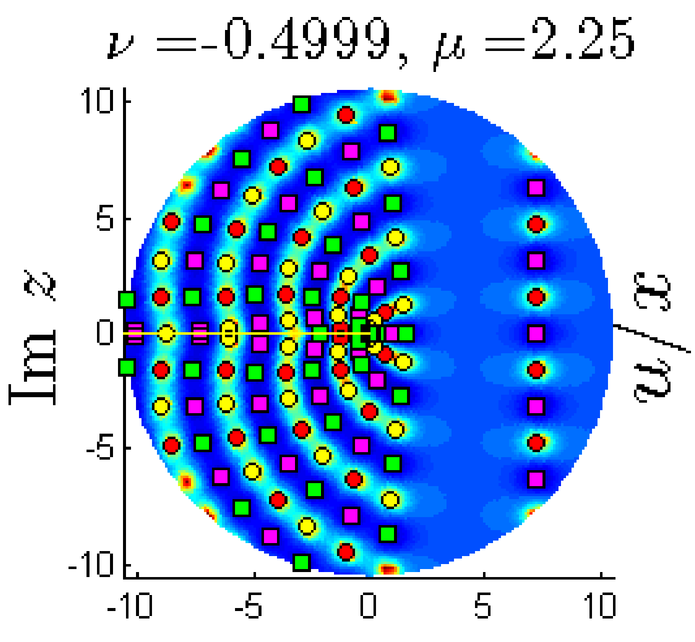

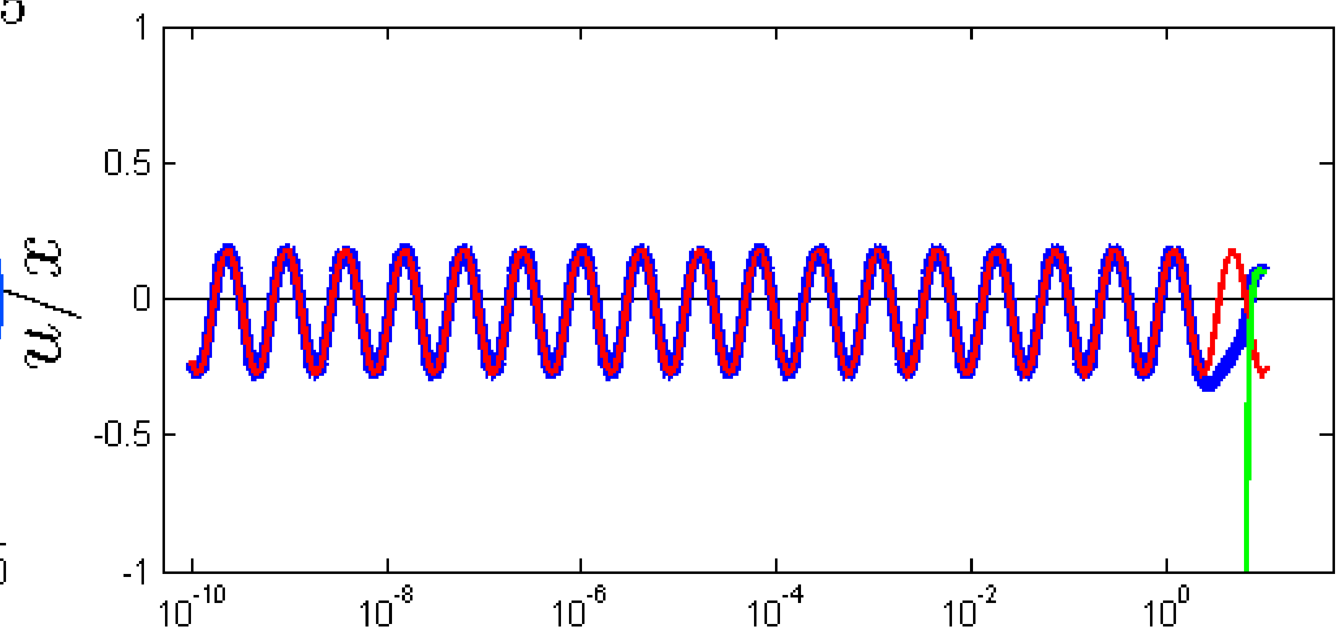

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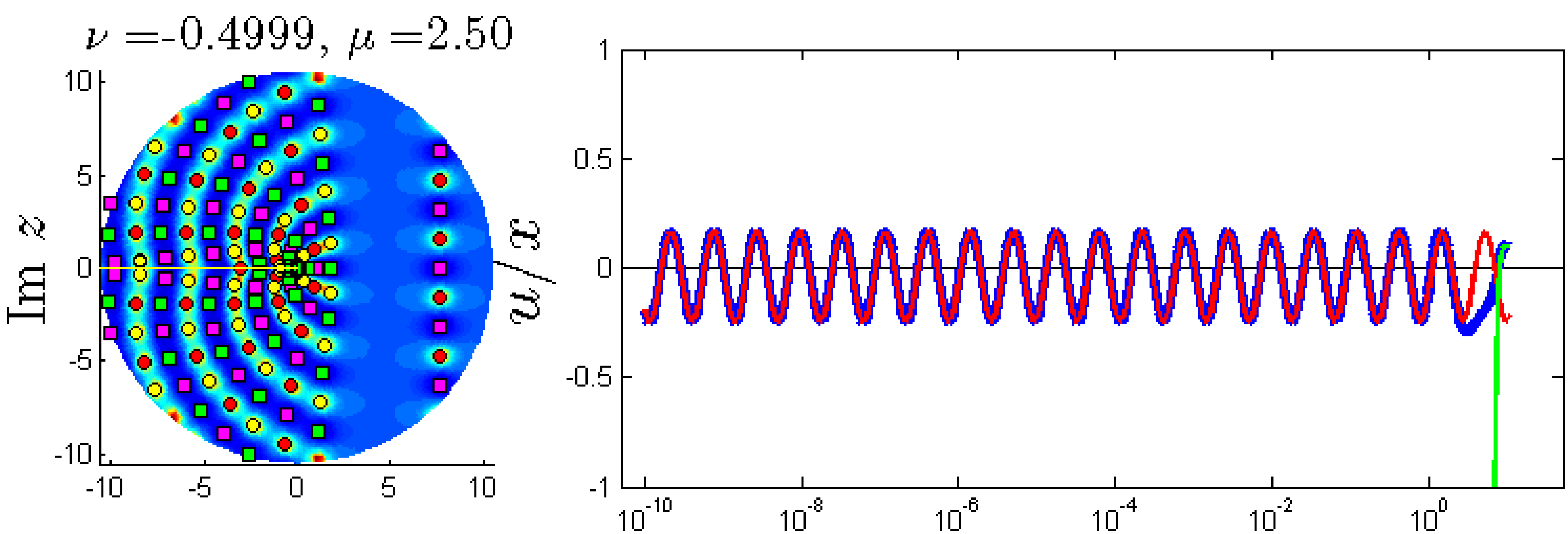


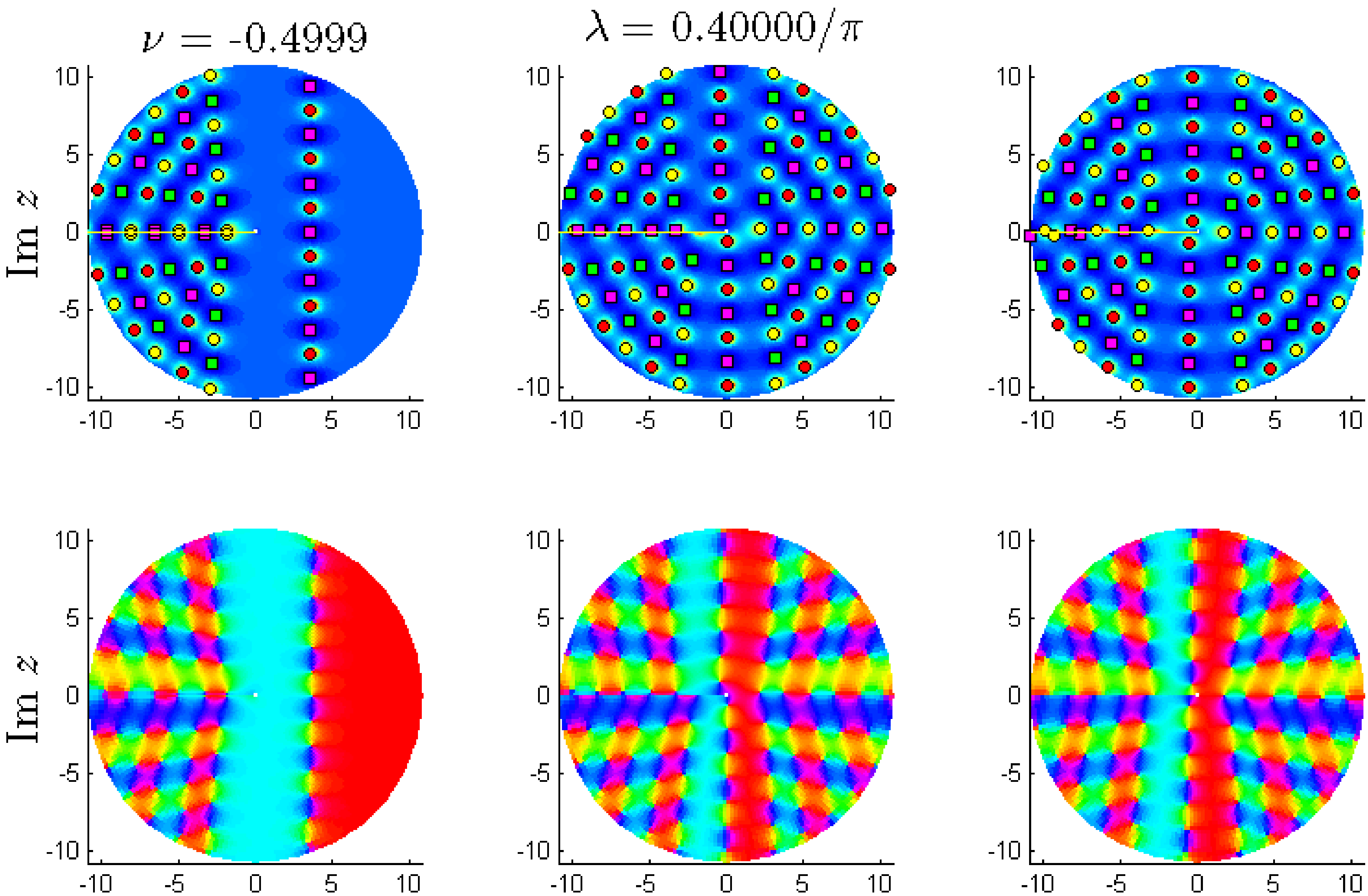
Tronquée P_{III} solutions

As $\boldsymbol{\lambda}$ is varied, multiple tronquée solutions occur on sheets other than the main sheet of the **MTW solutions**

Lin, Dai and Tibboel [2014] proved the existence of tronquée P_{III} solutions whose pole-free sectors have angular widths of

- $\boldsymbol{\pi}$ and $\boldsymbol{2\pi}$ if $\gamma = 1$ and $\boldsymbol{\delta} = -1$, and
- $\boldsymbol{\frac{3\pi}{2}}$ and $\boldsymbol{3\pi}$ if $\boldsymbol{\alpha} = 1, \gamma = 0$ and $\boldsymbol{\delta} = -1$.

Tronquée P_{III} solutions



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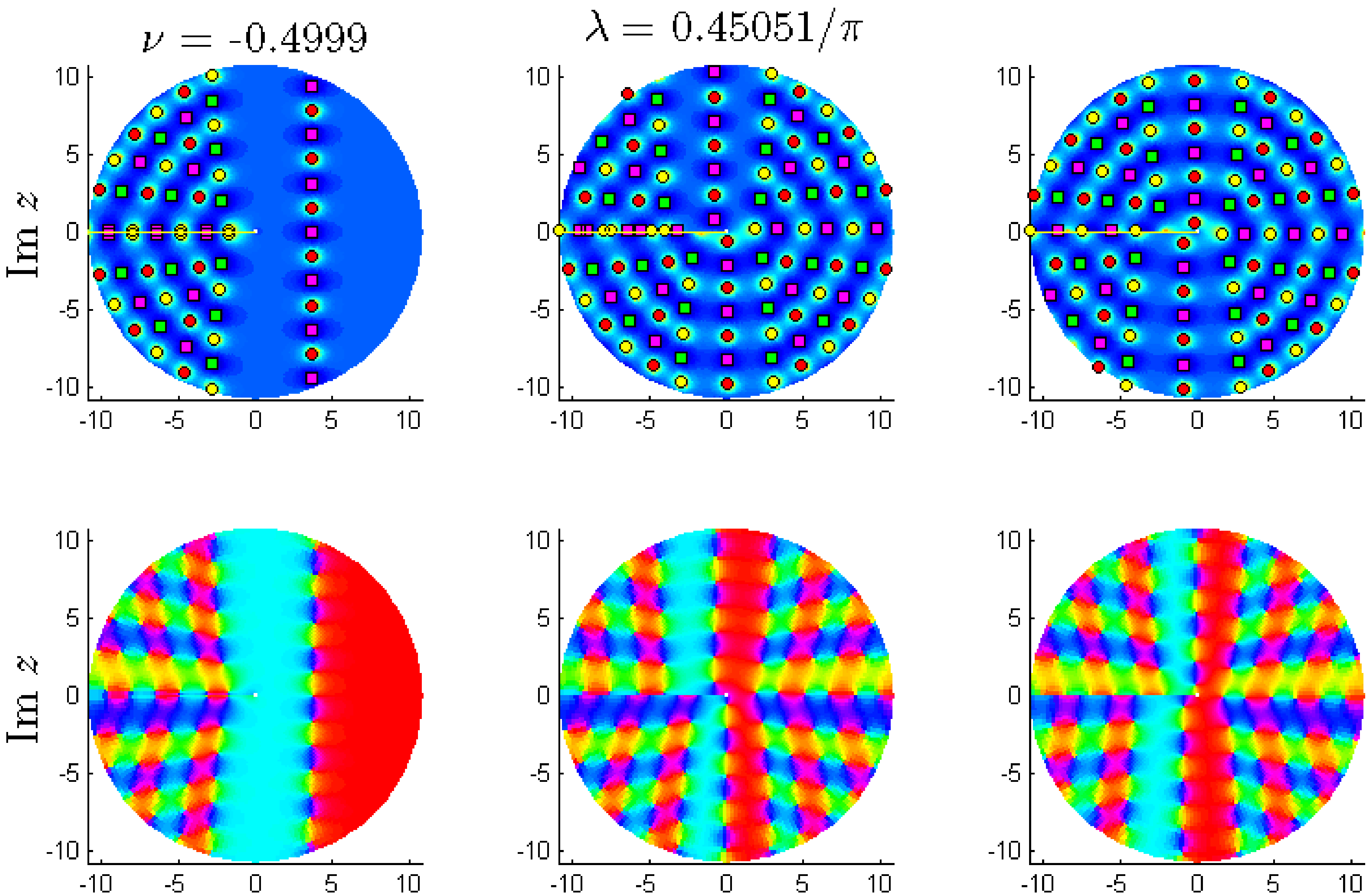

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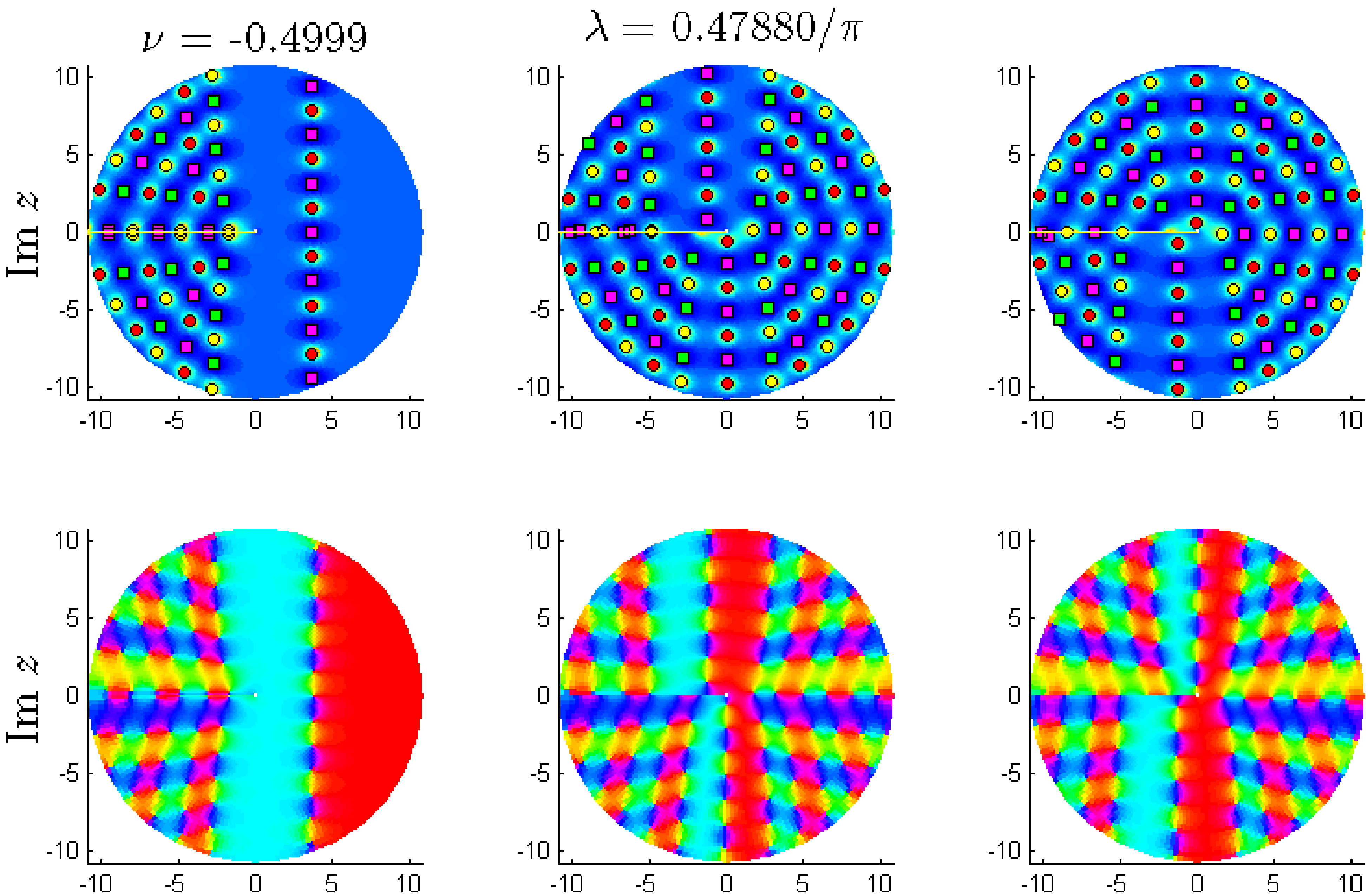

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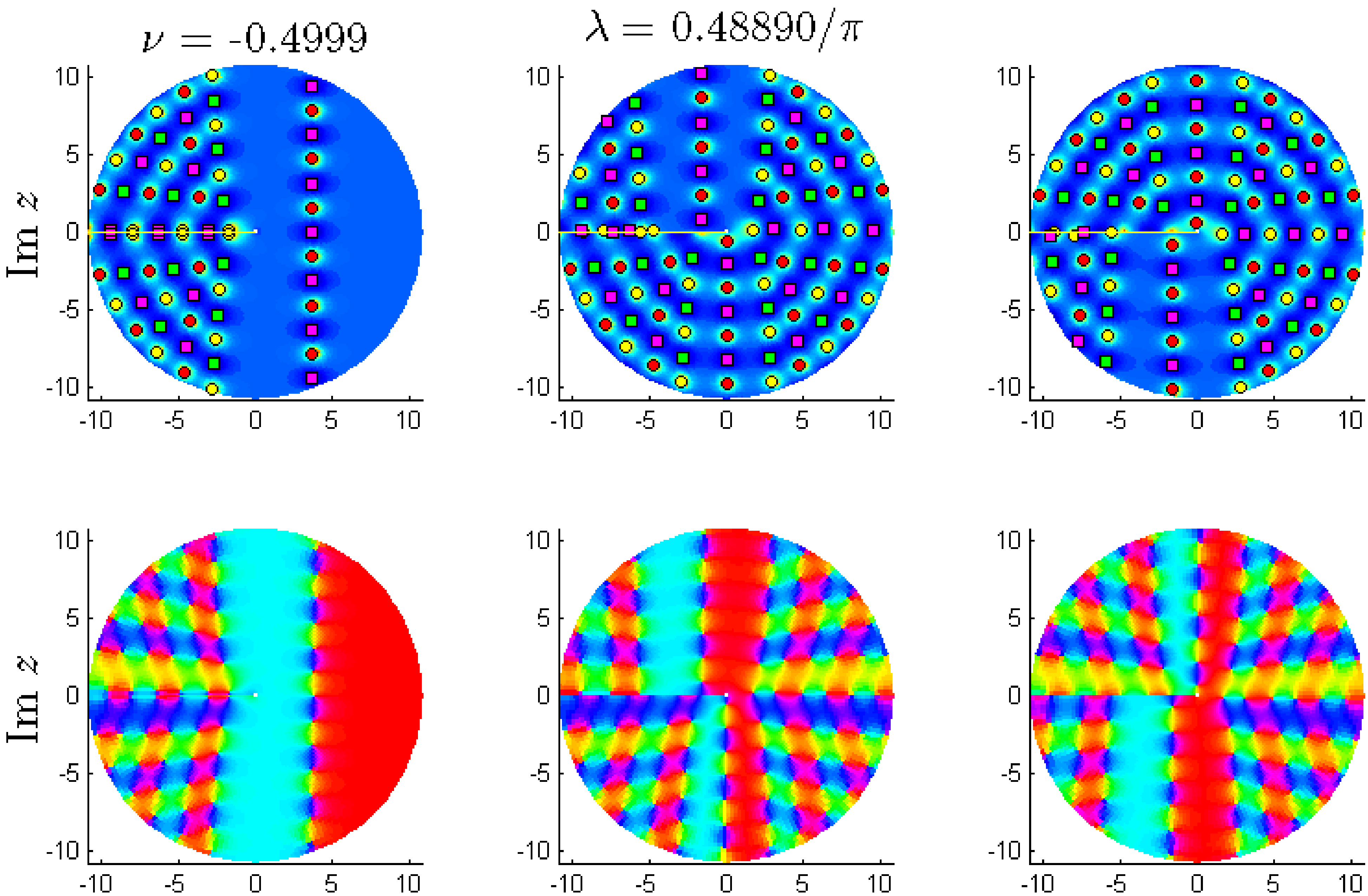

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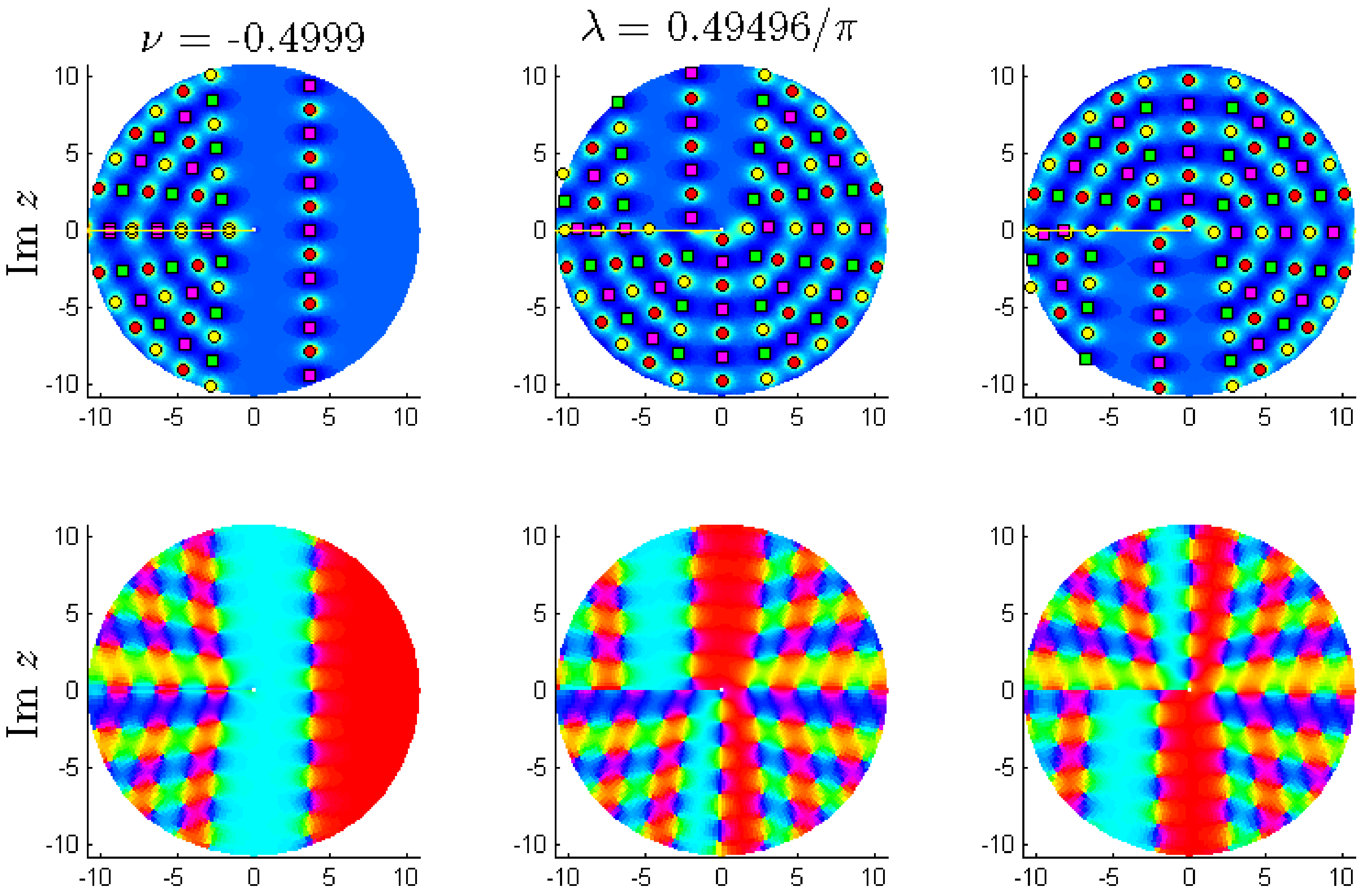

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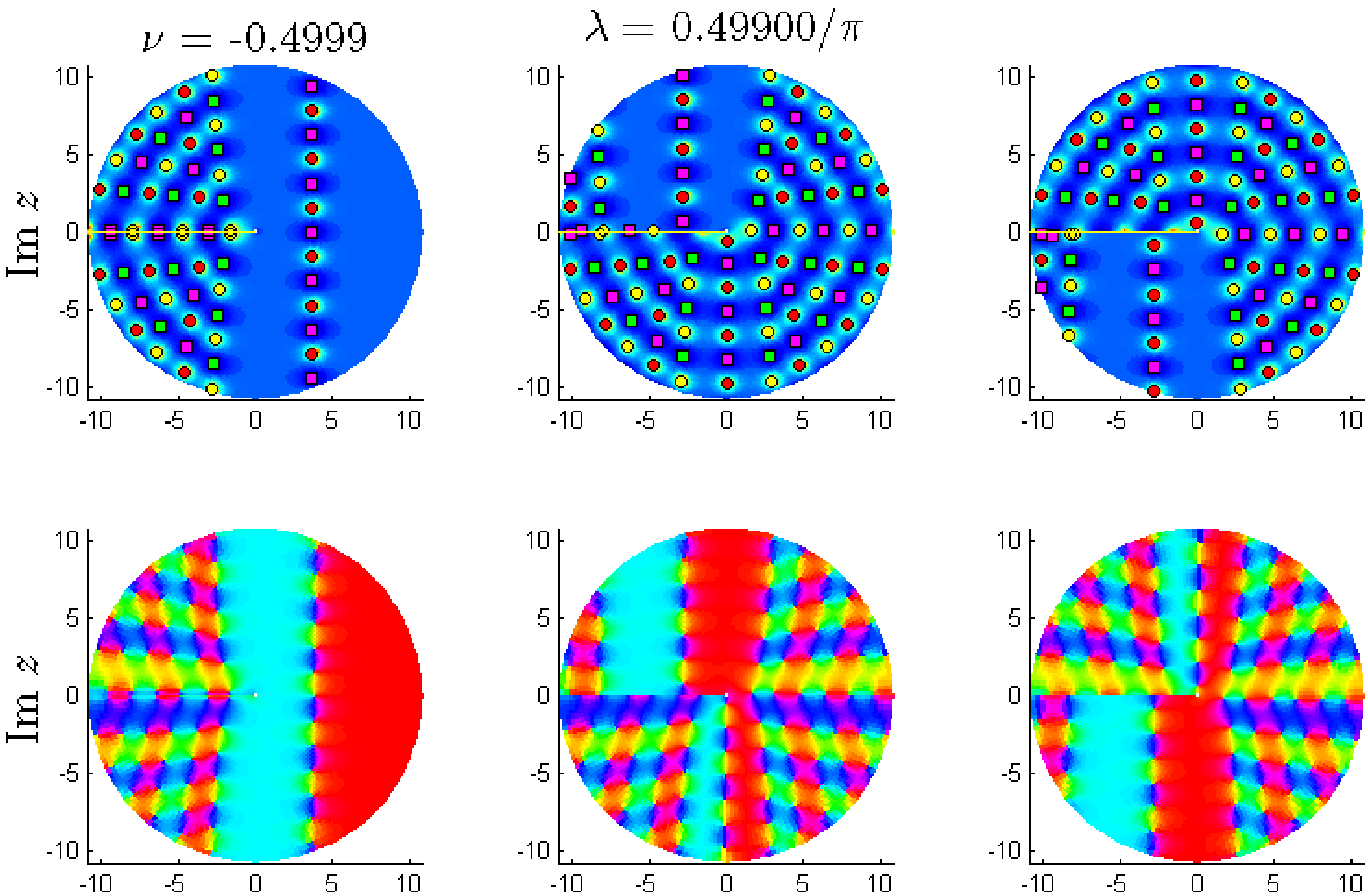

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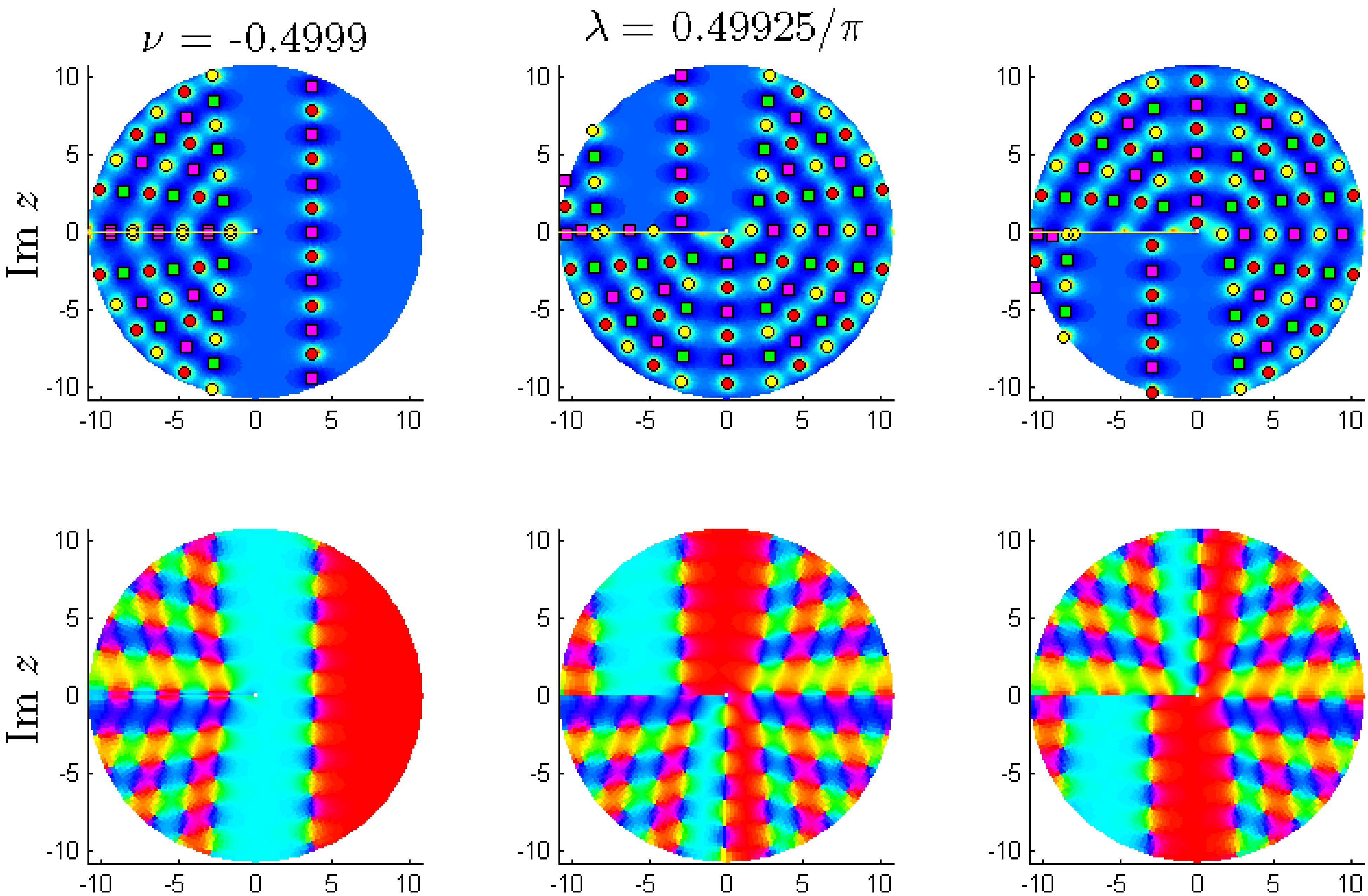

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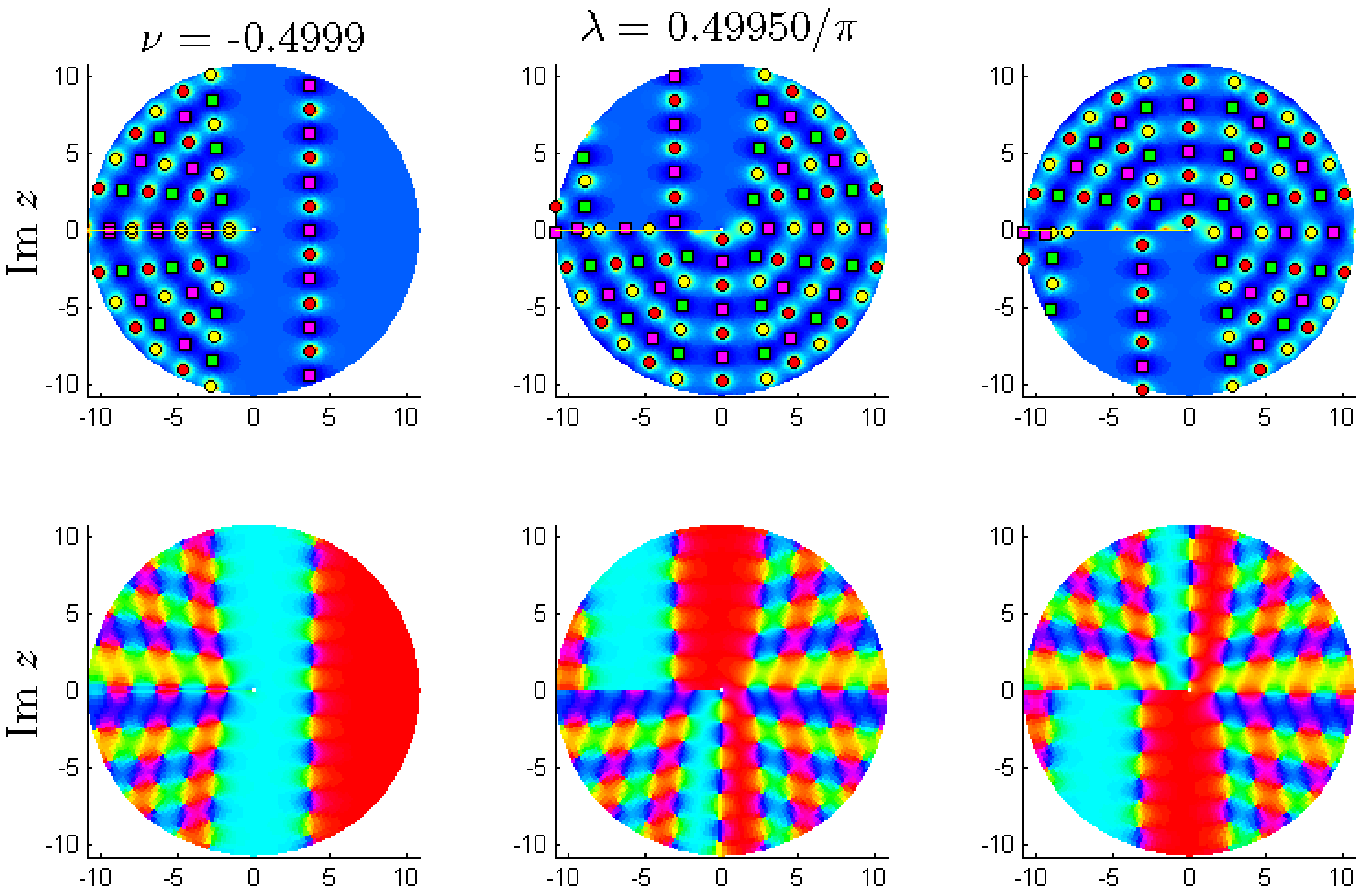

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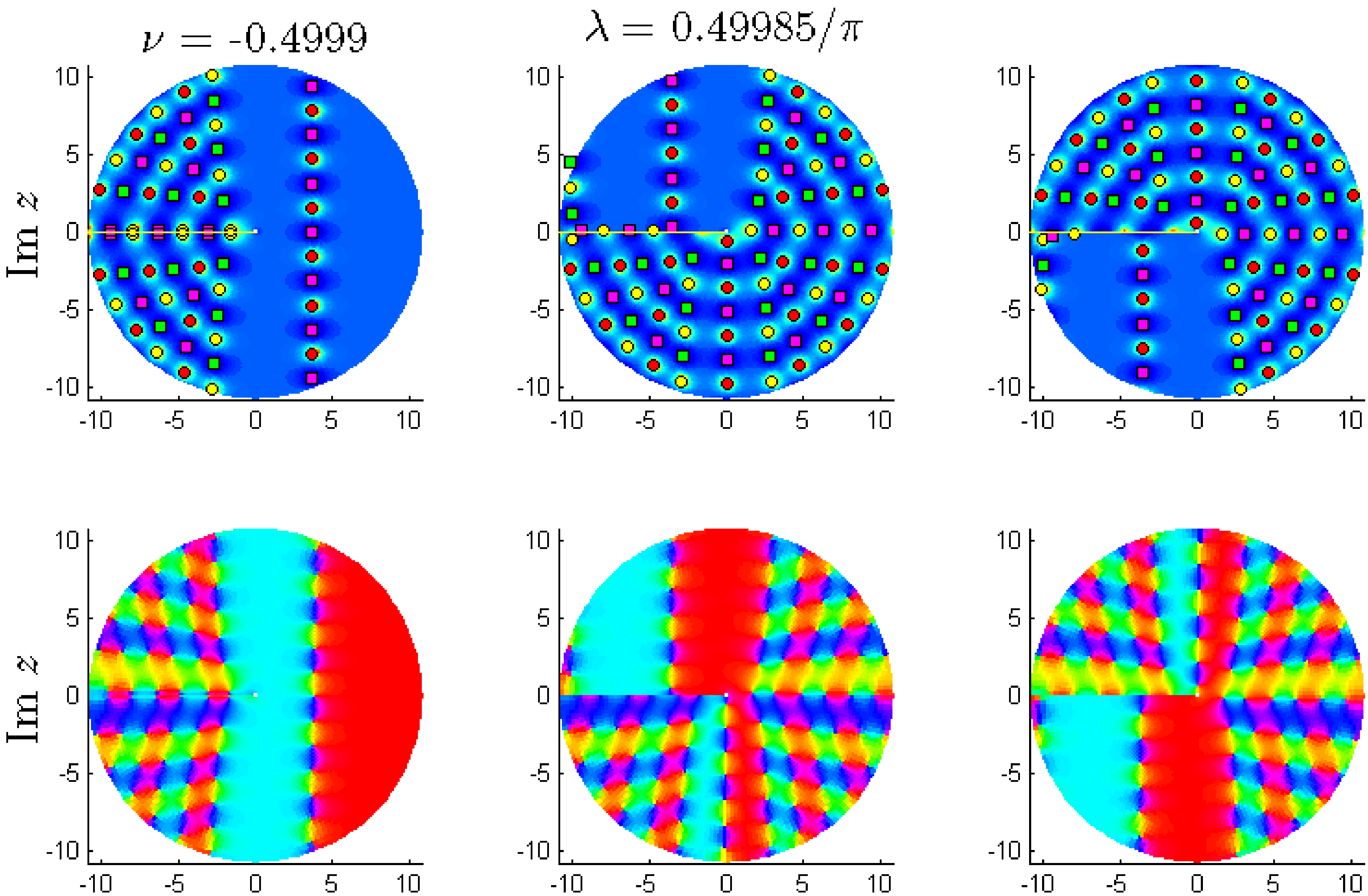

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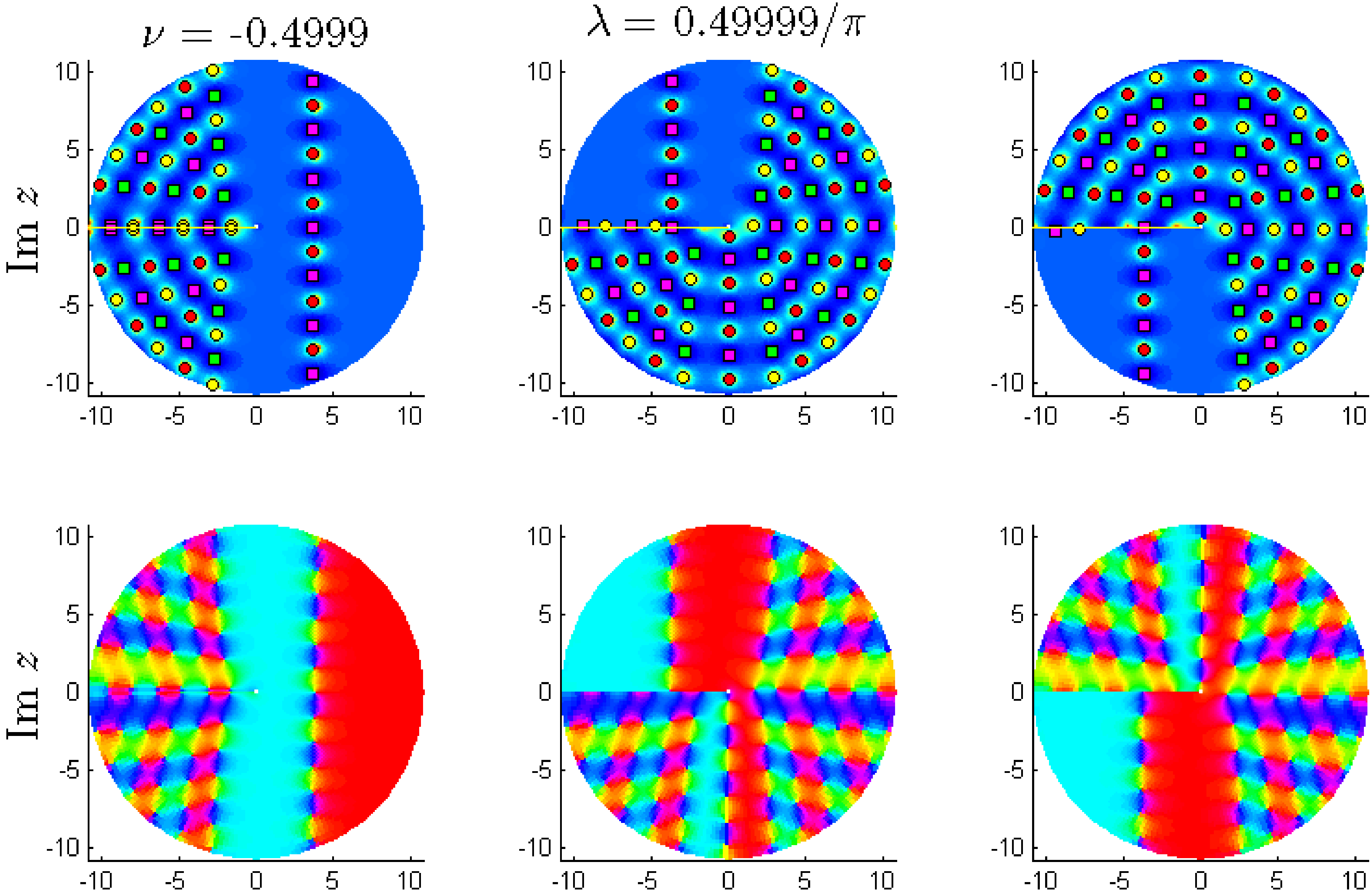

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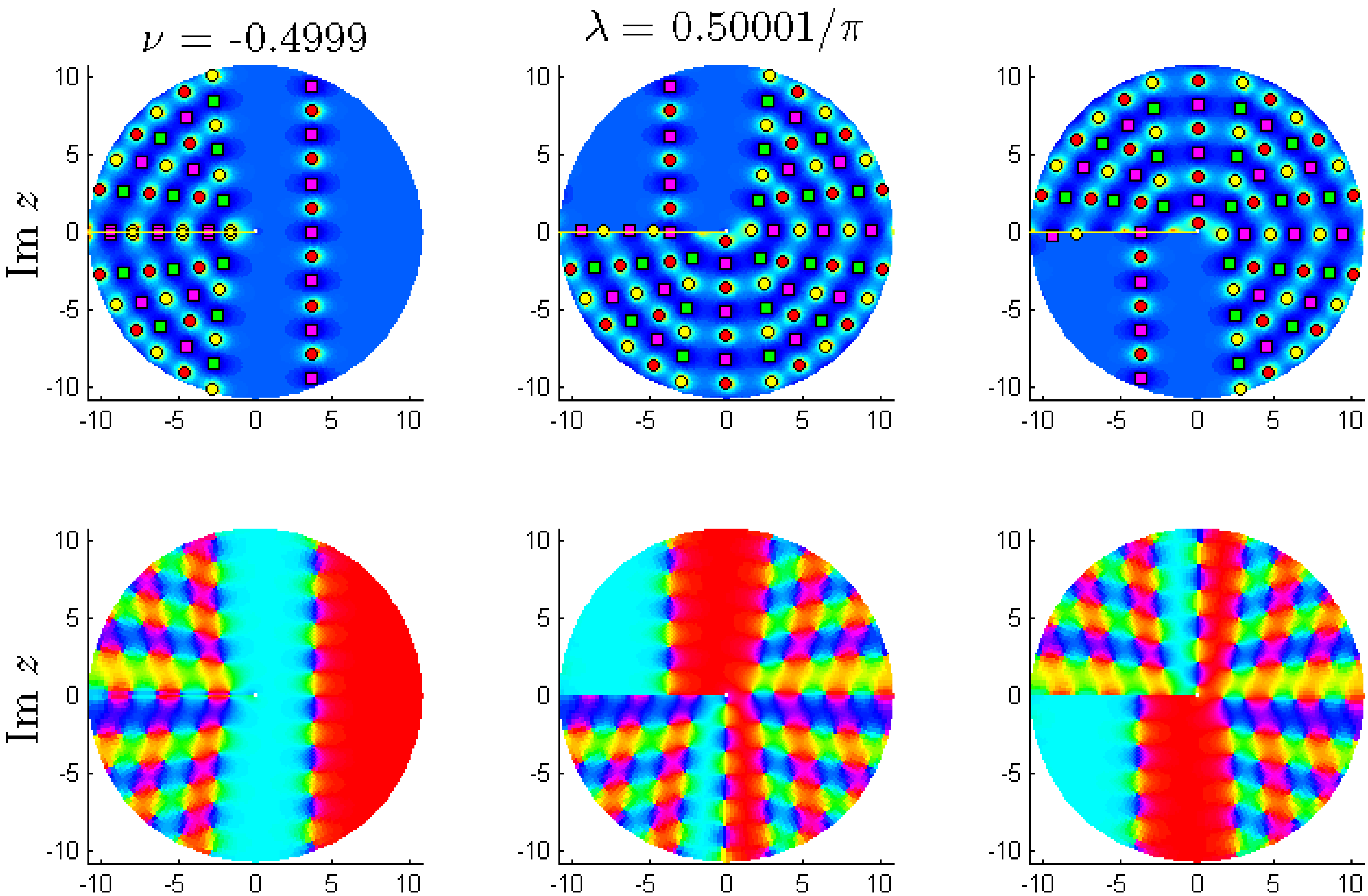

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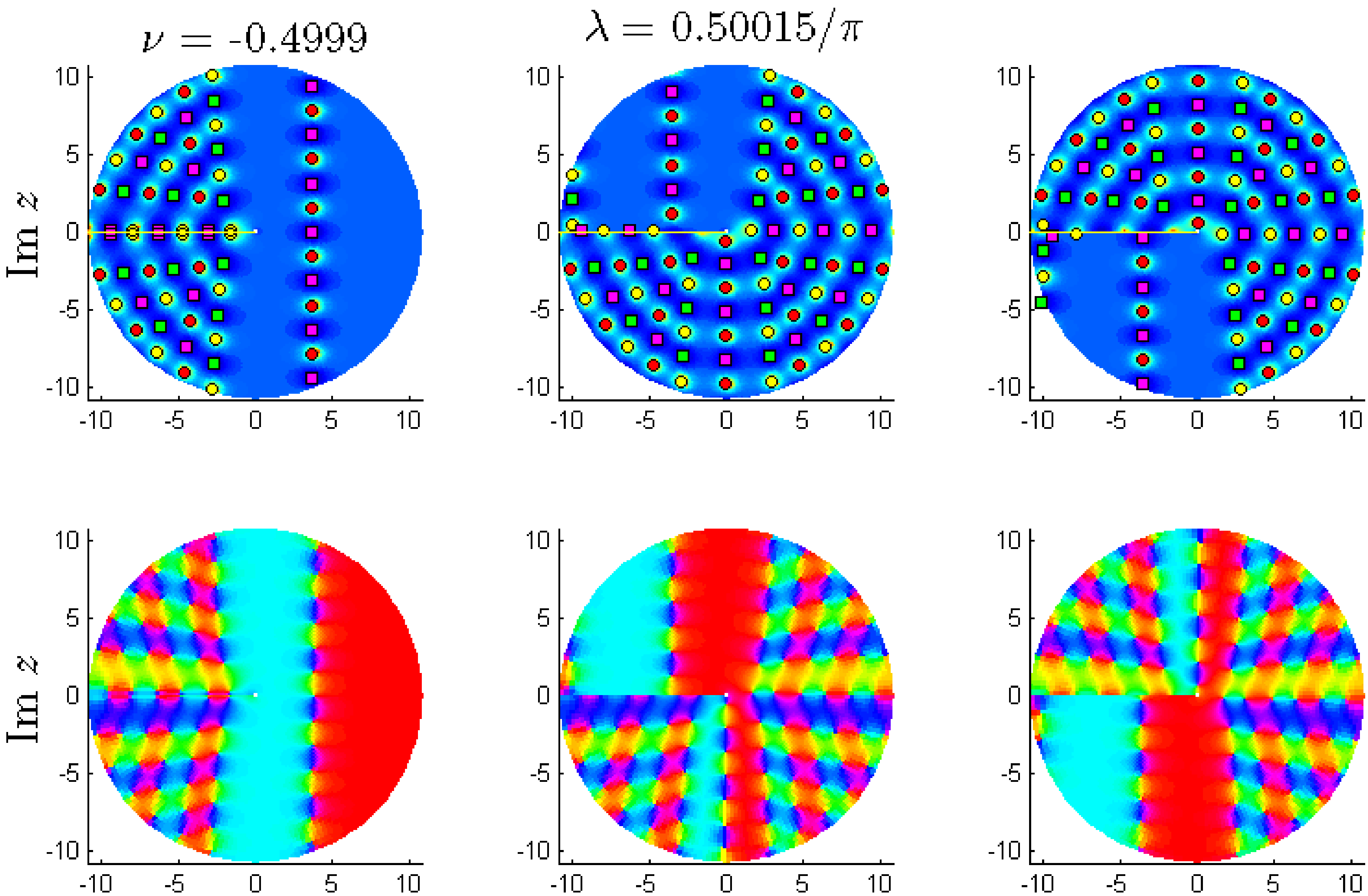

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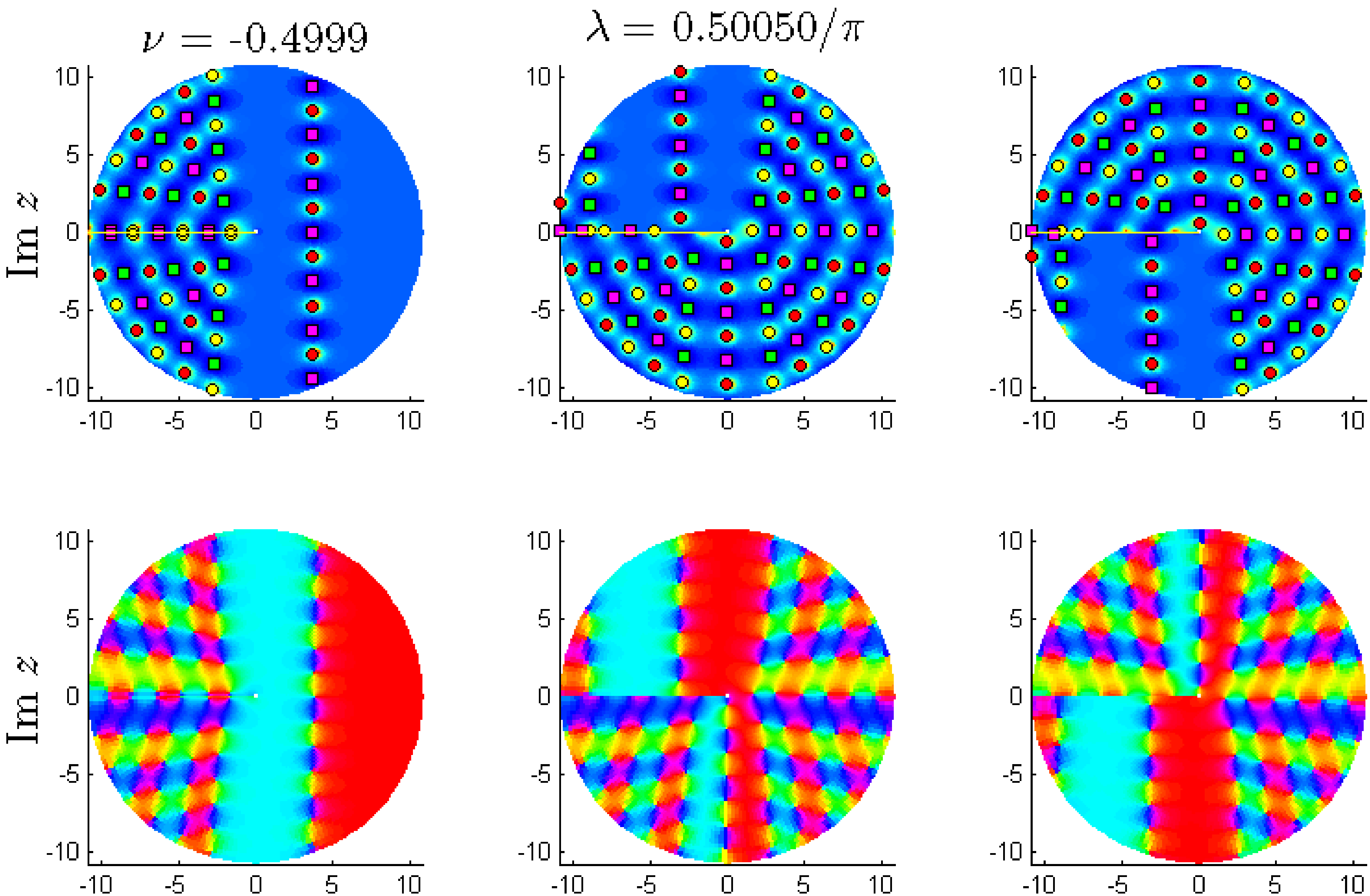

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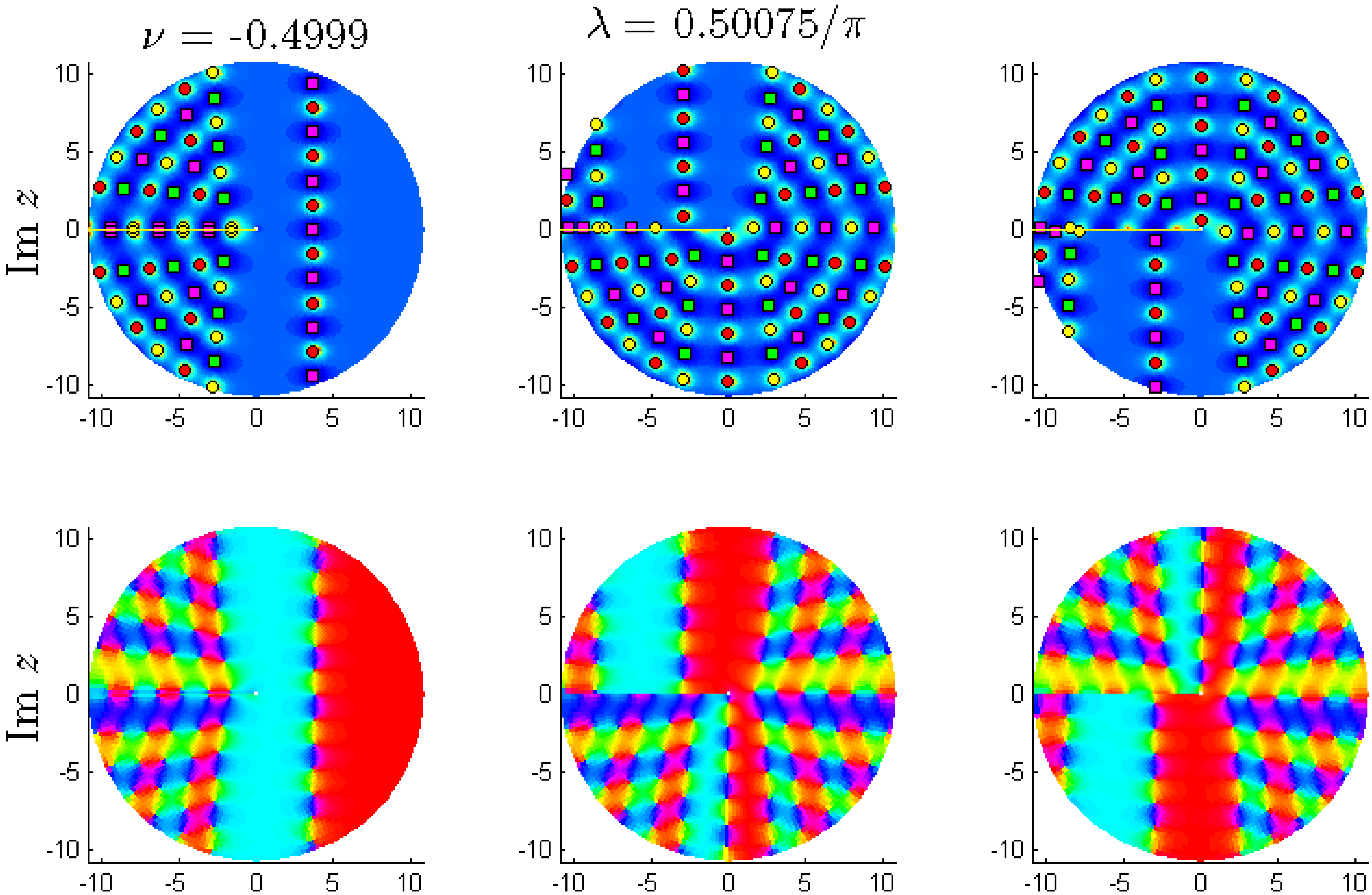

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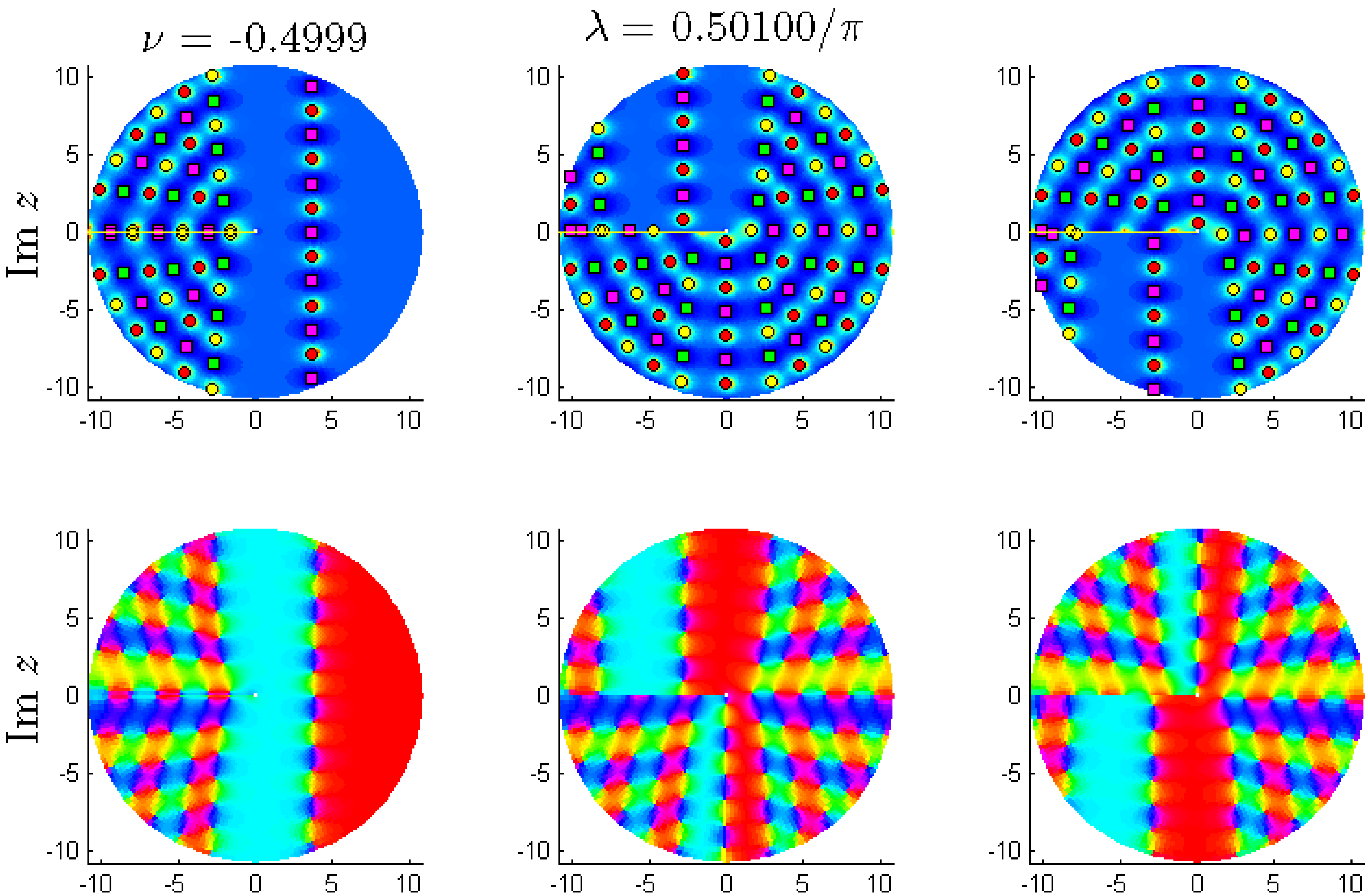

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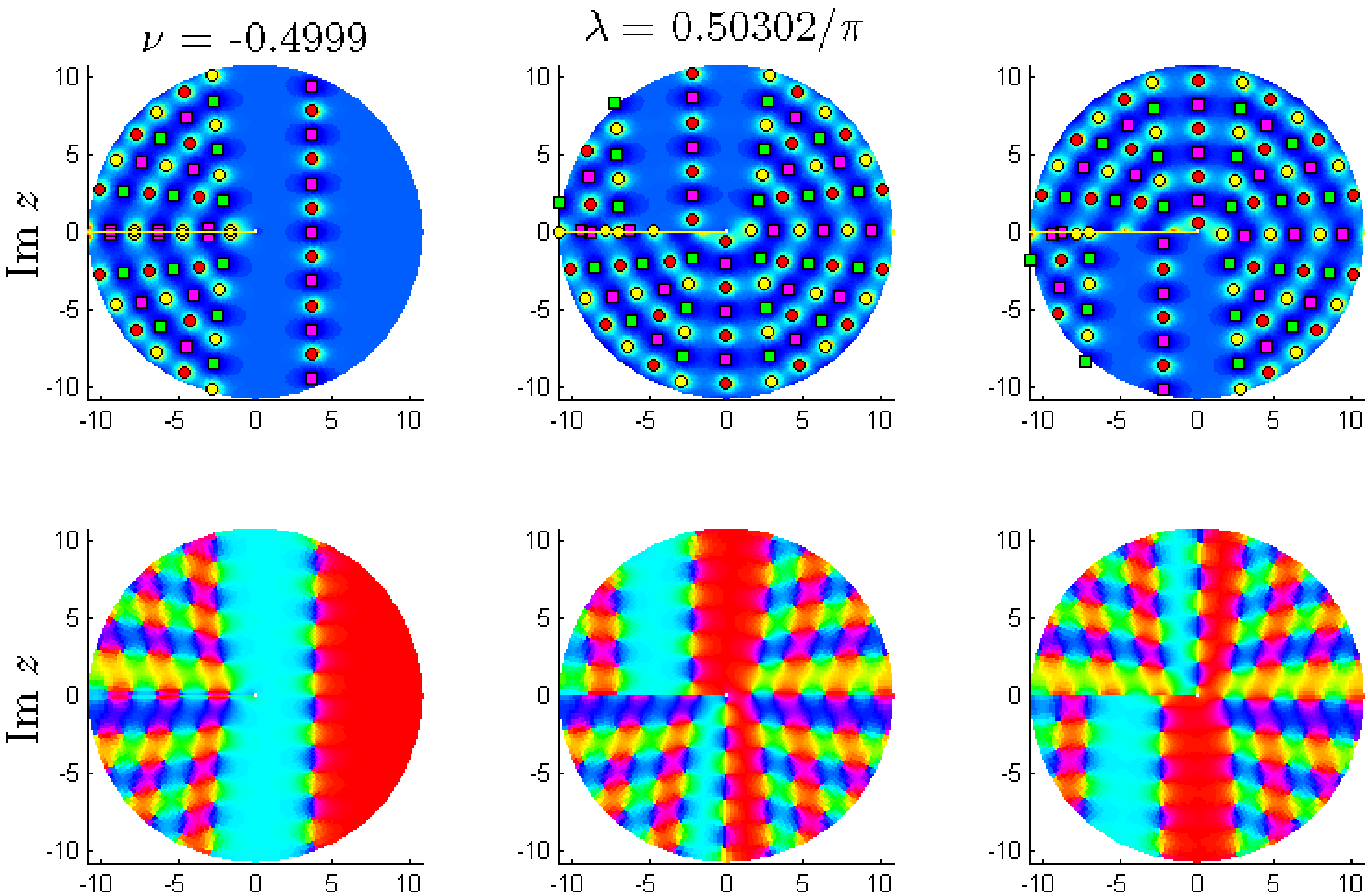

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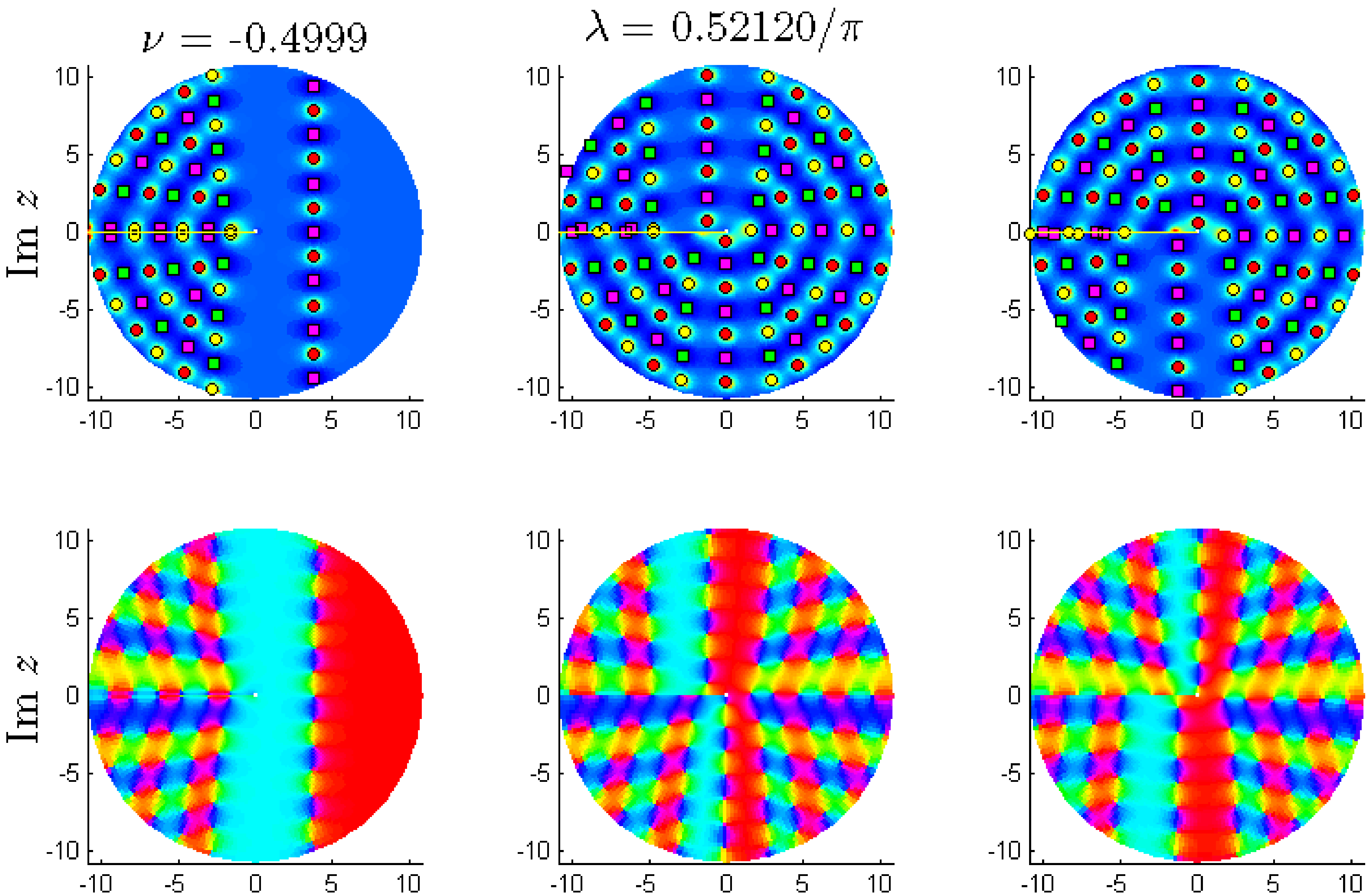

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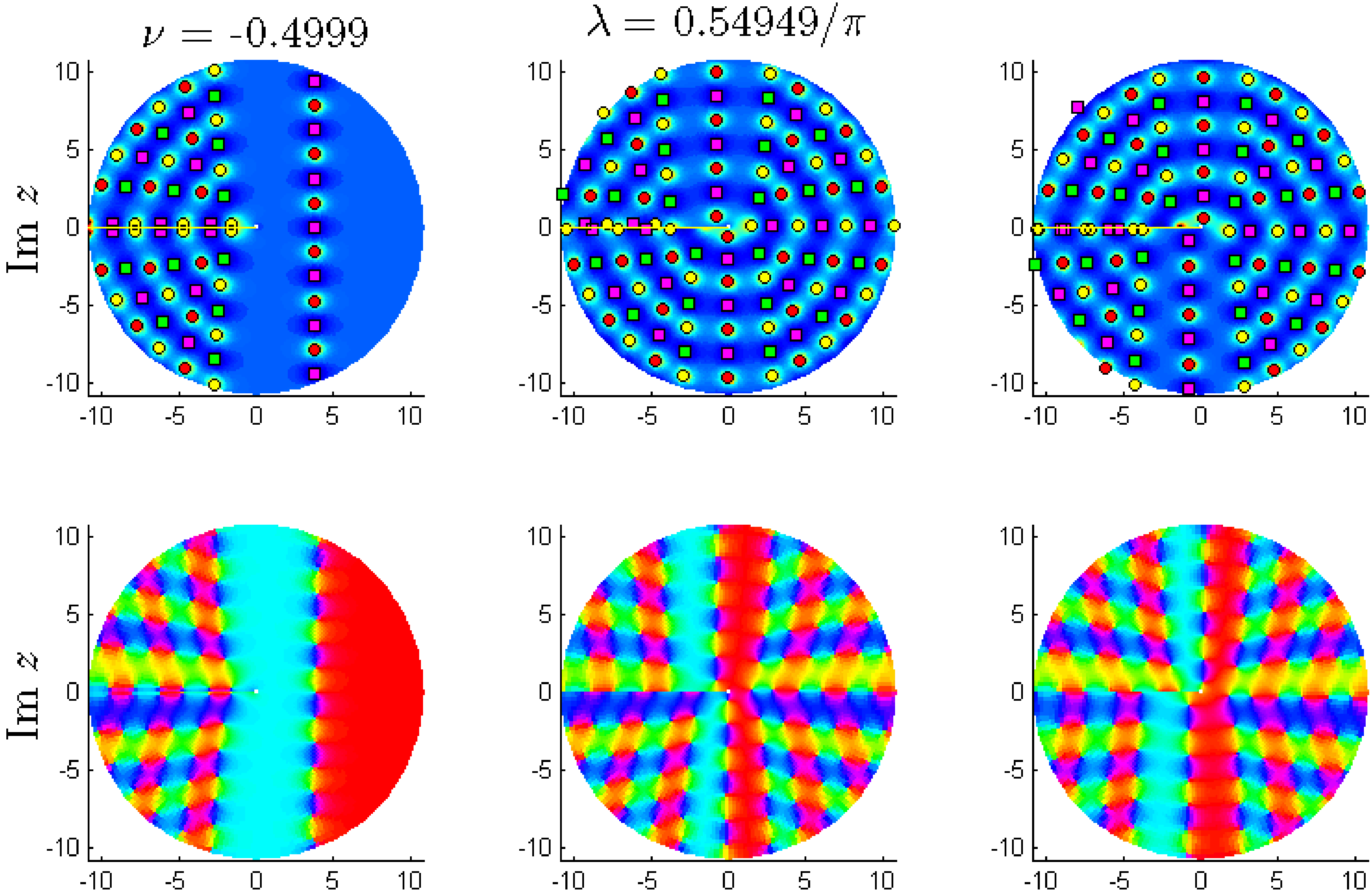

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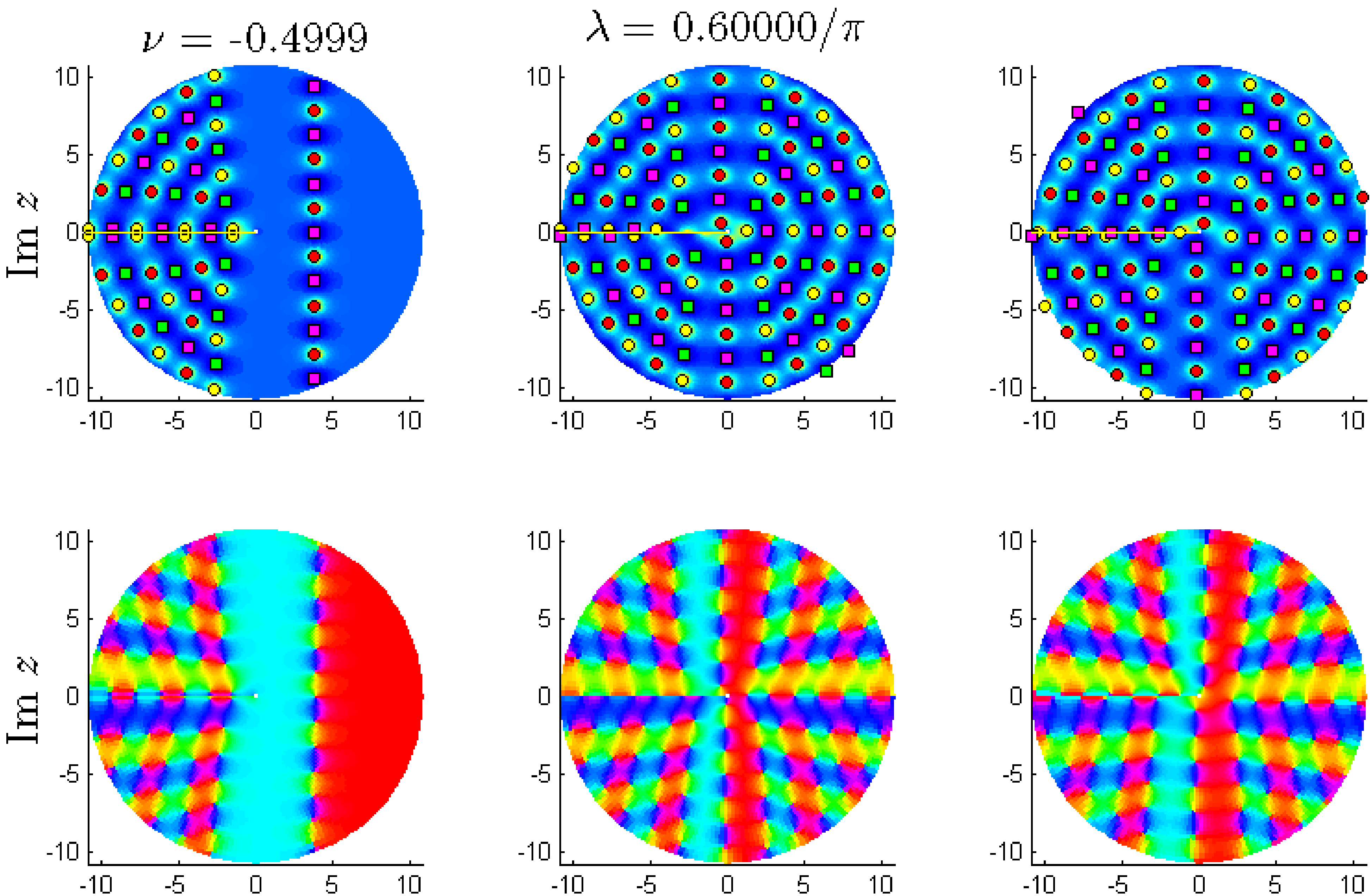

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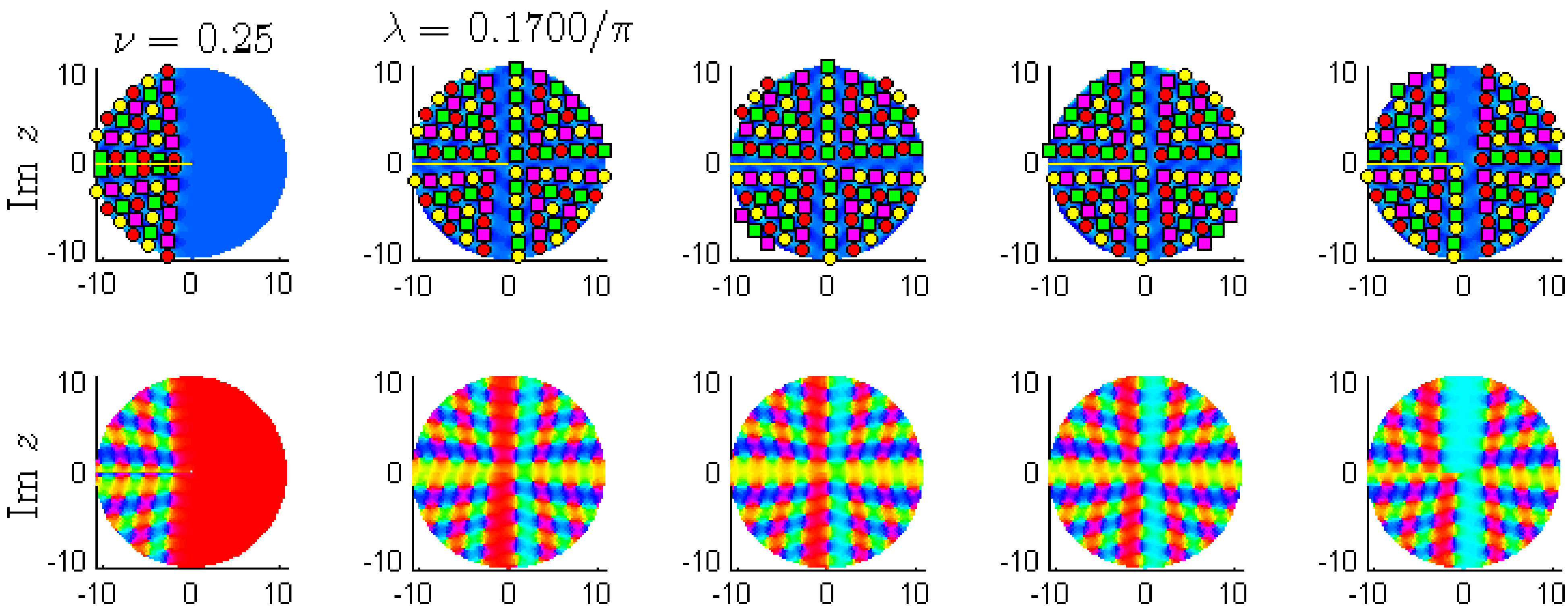

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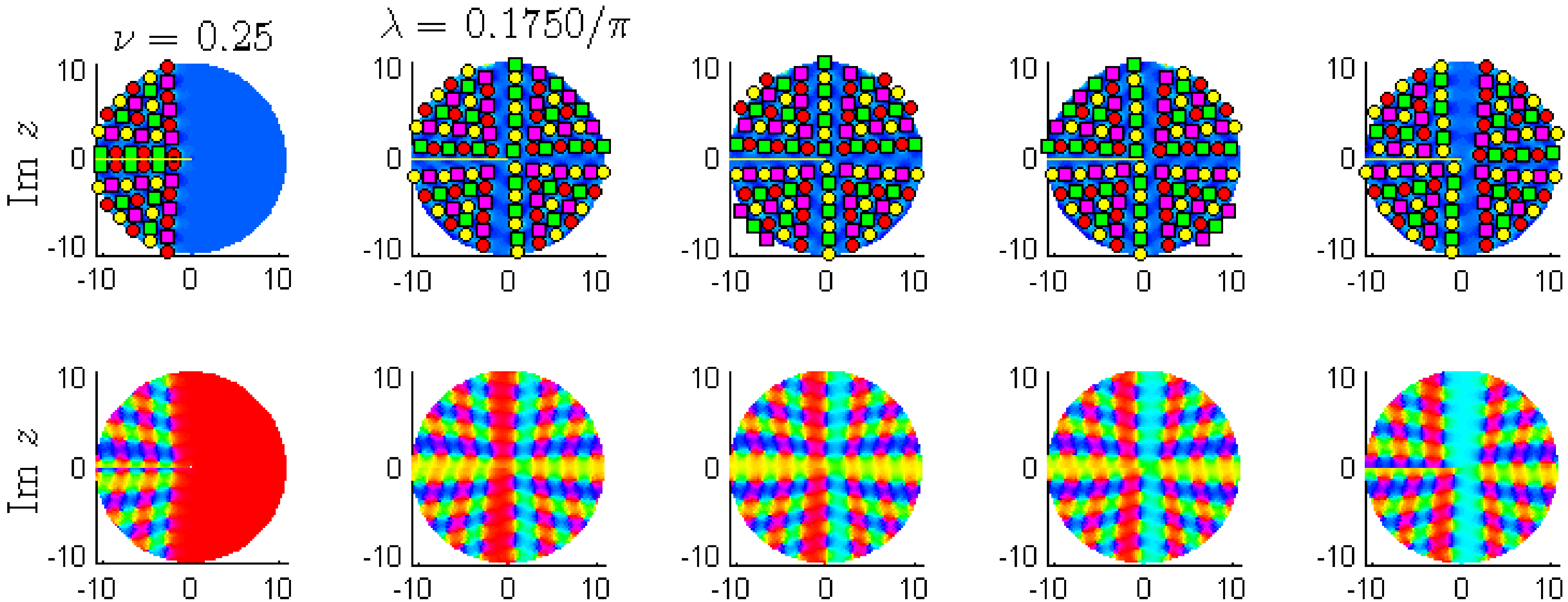

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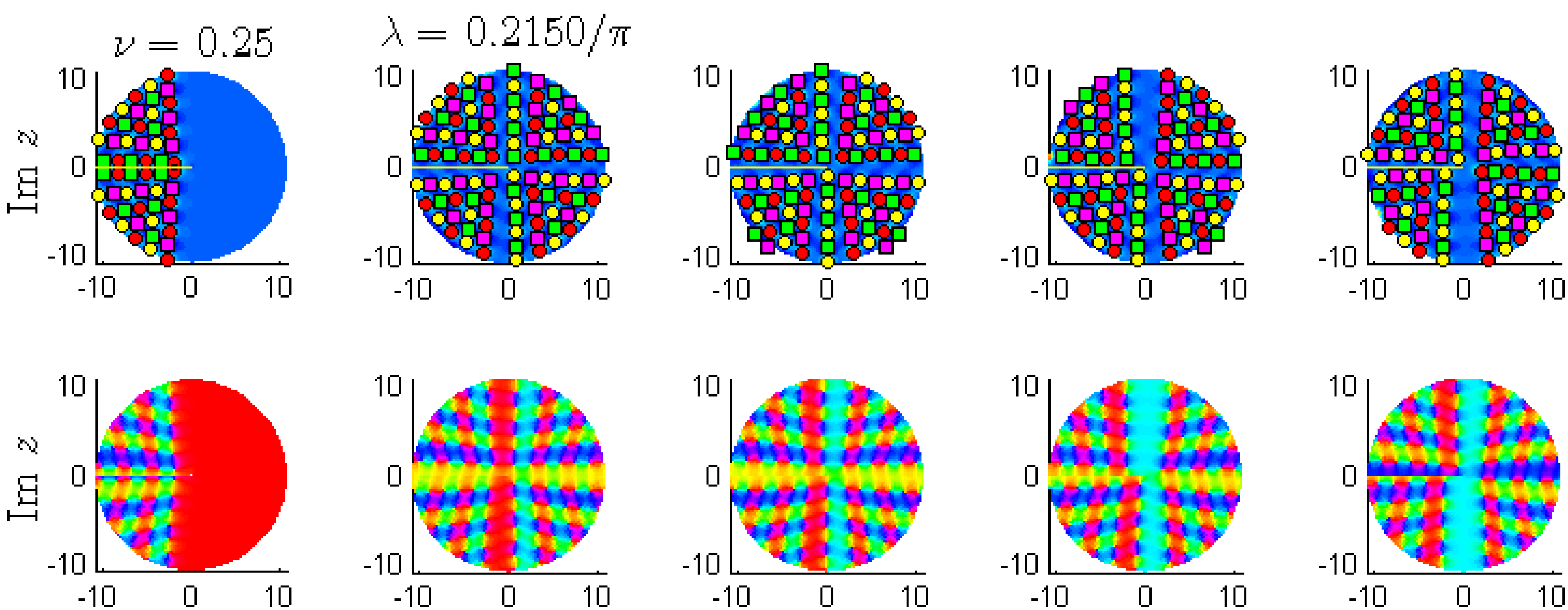

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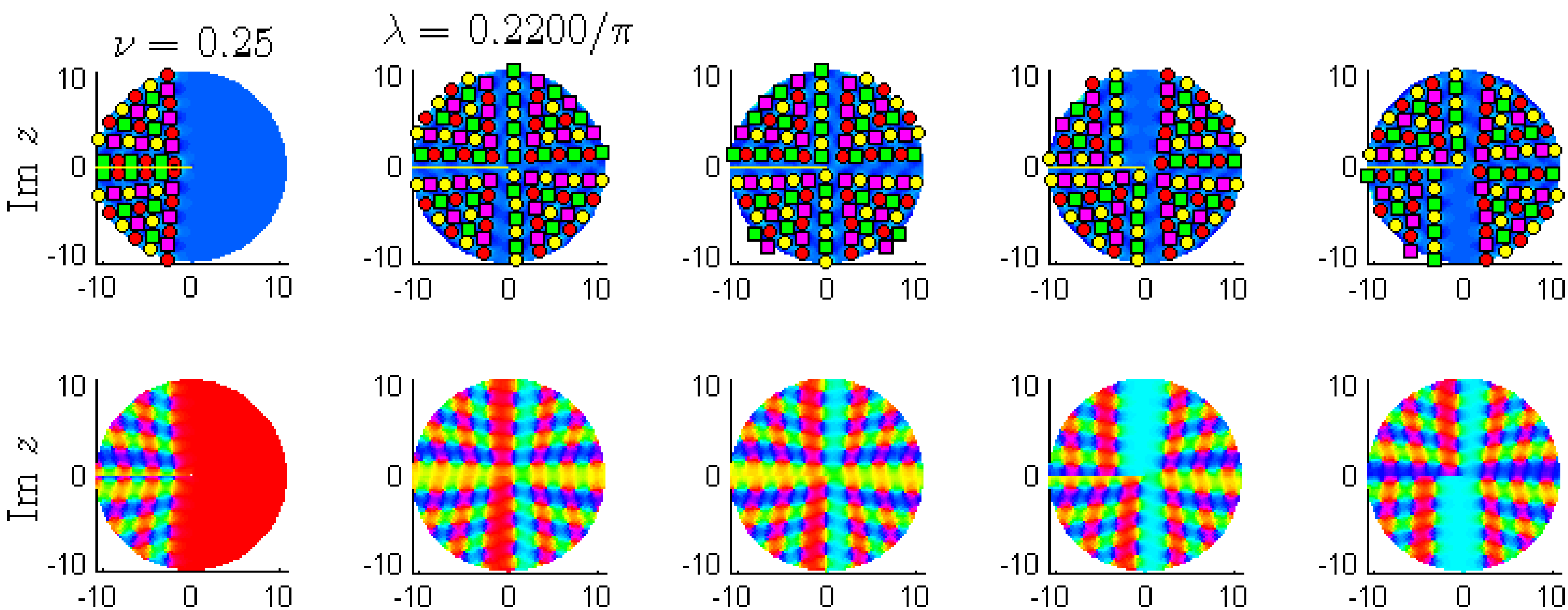

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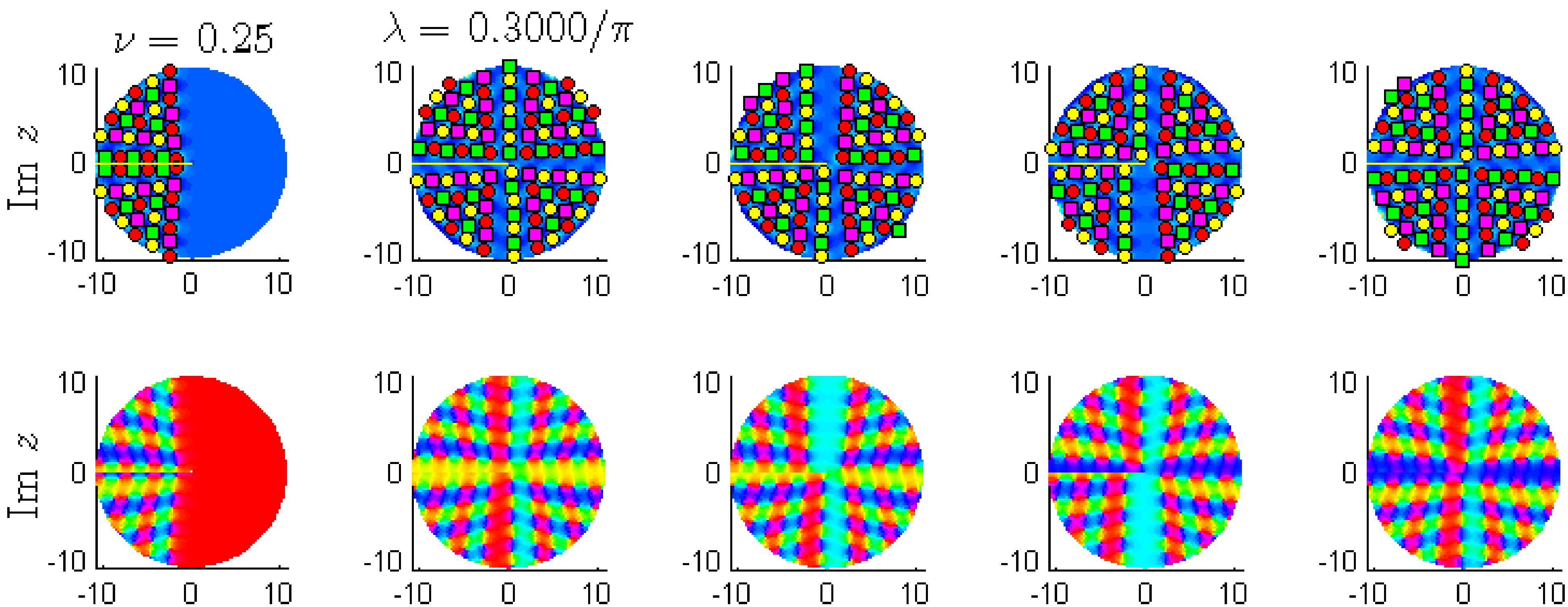

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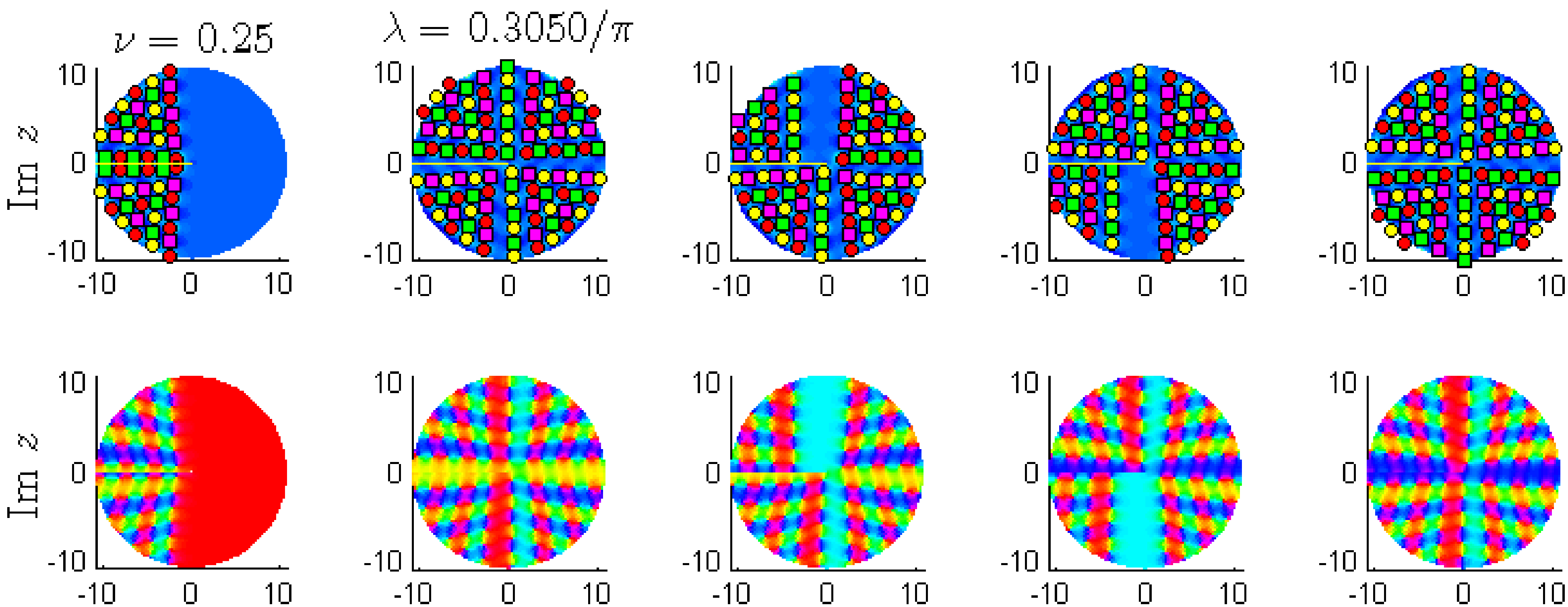

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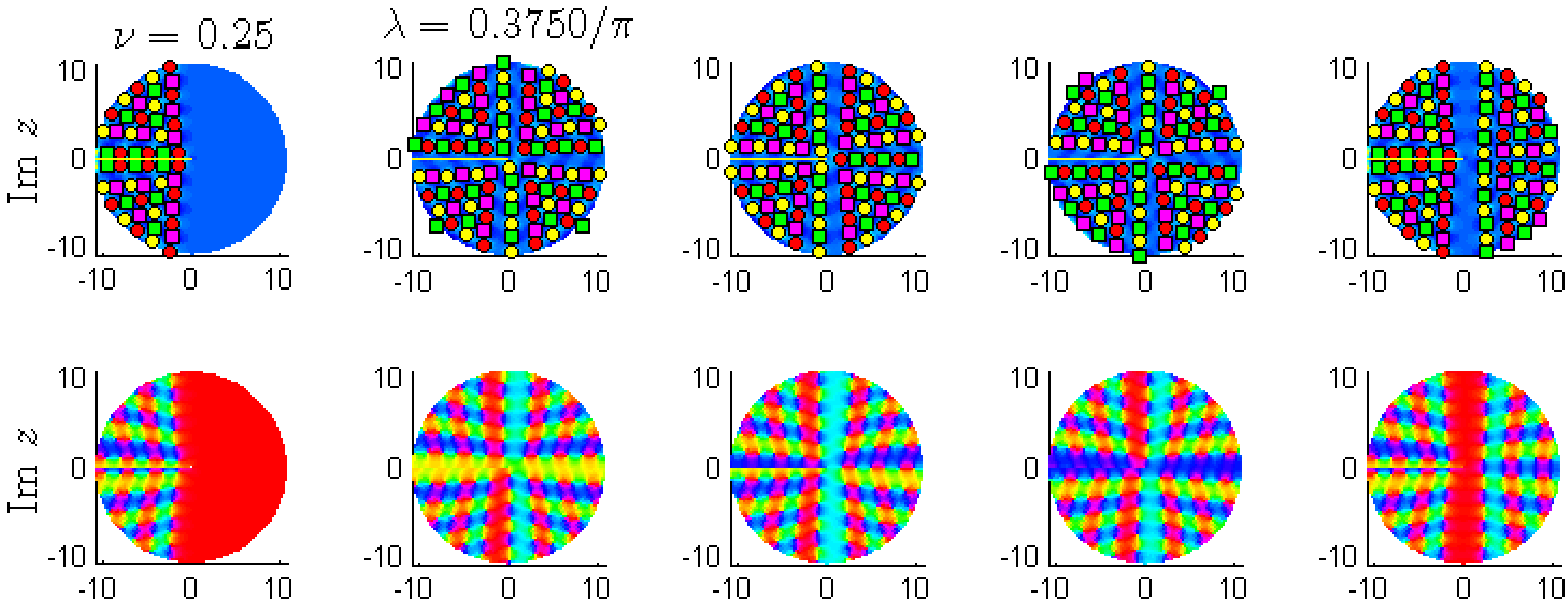

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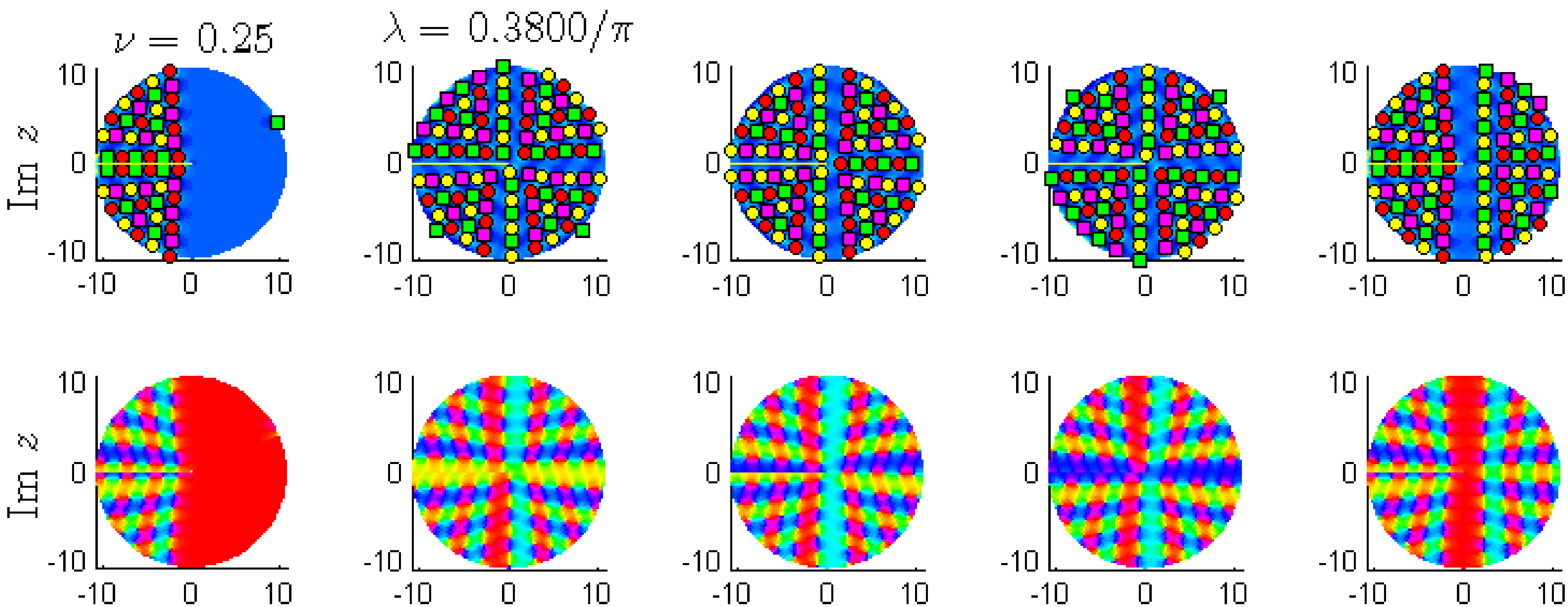

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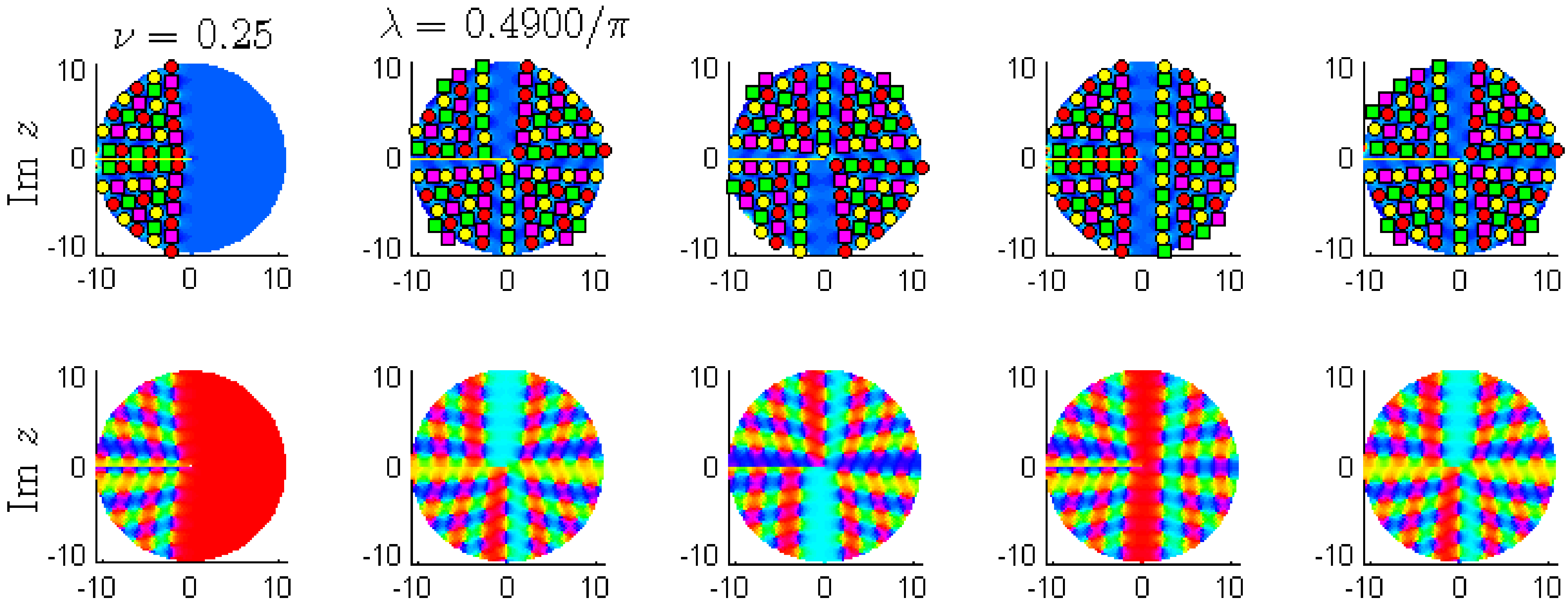

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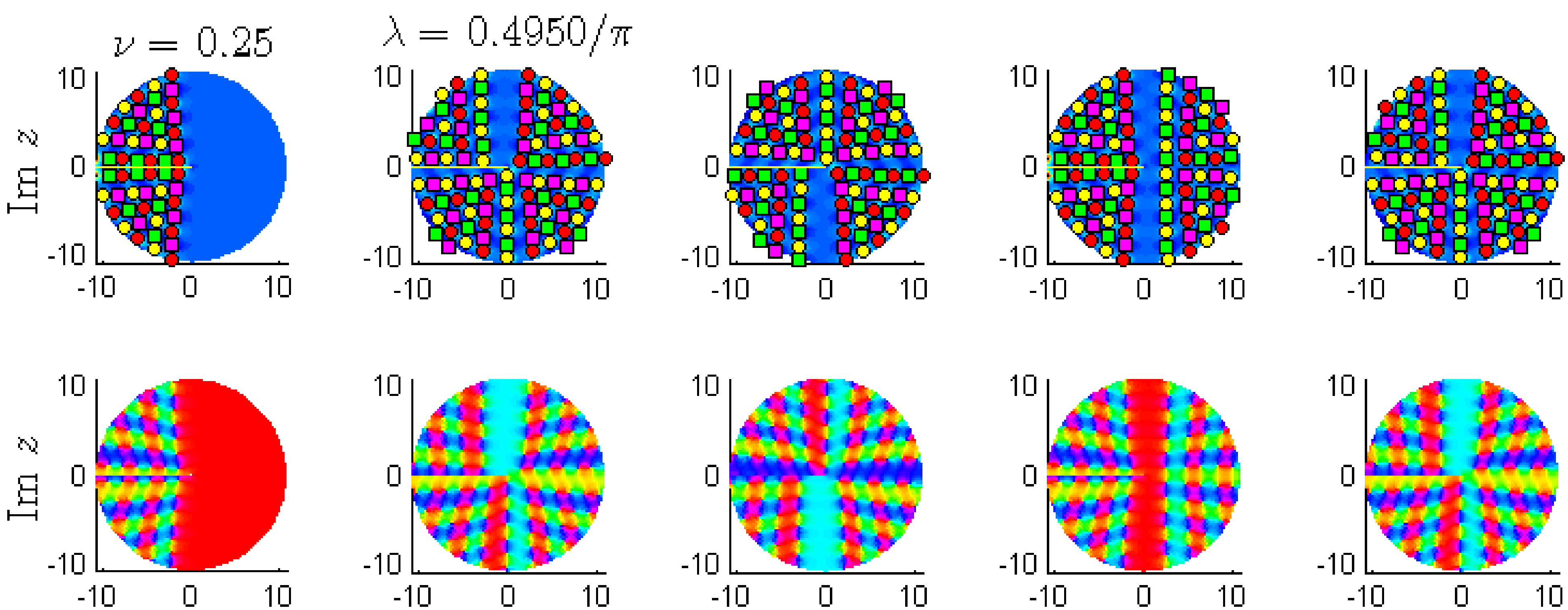

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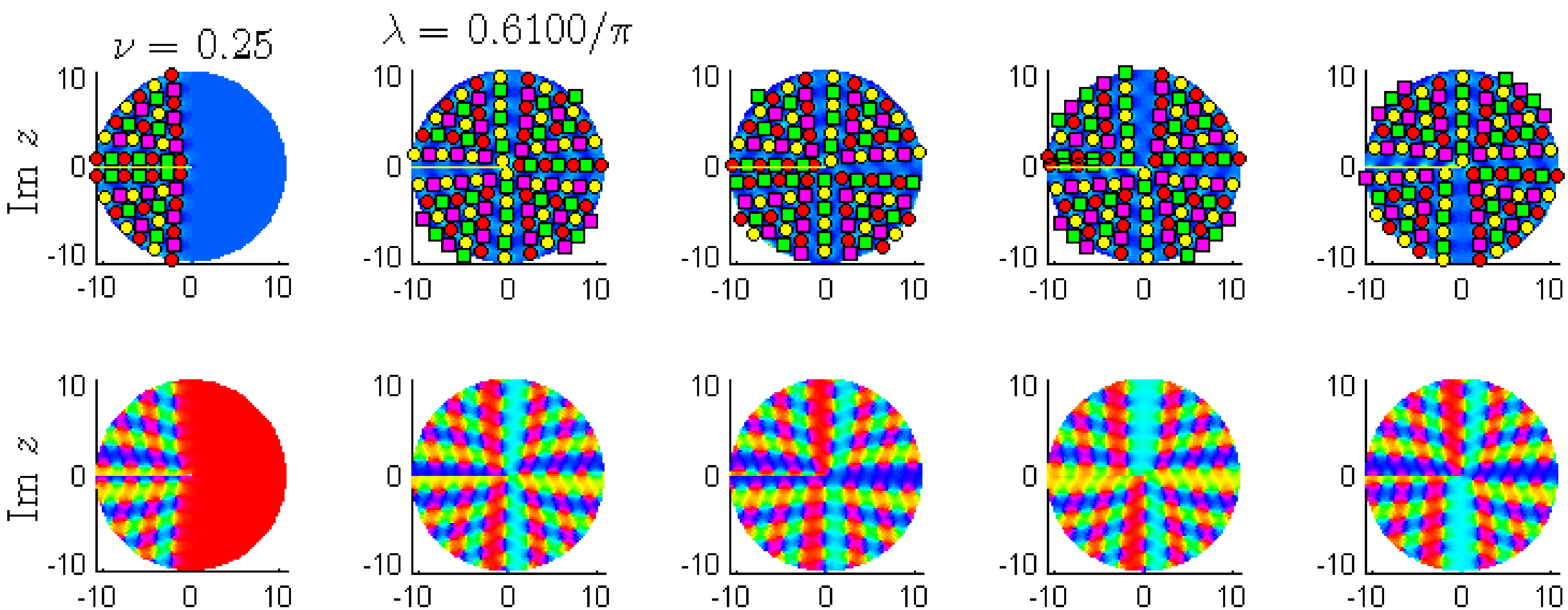

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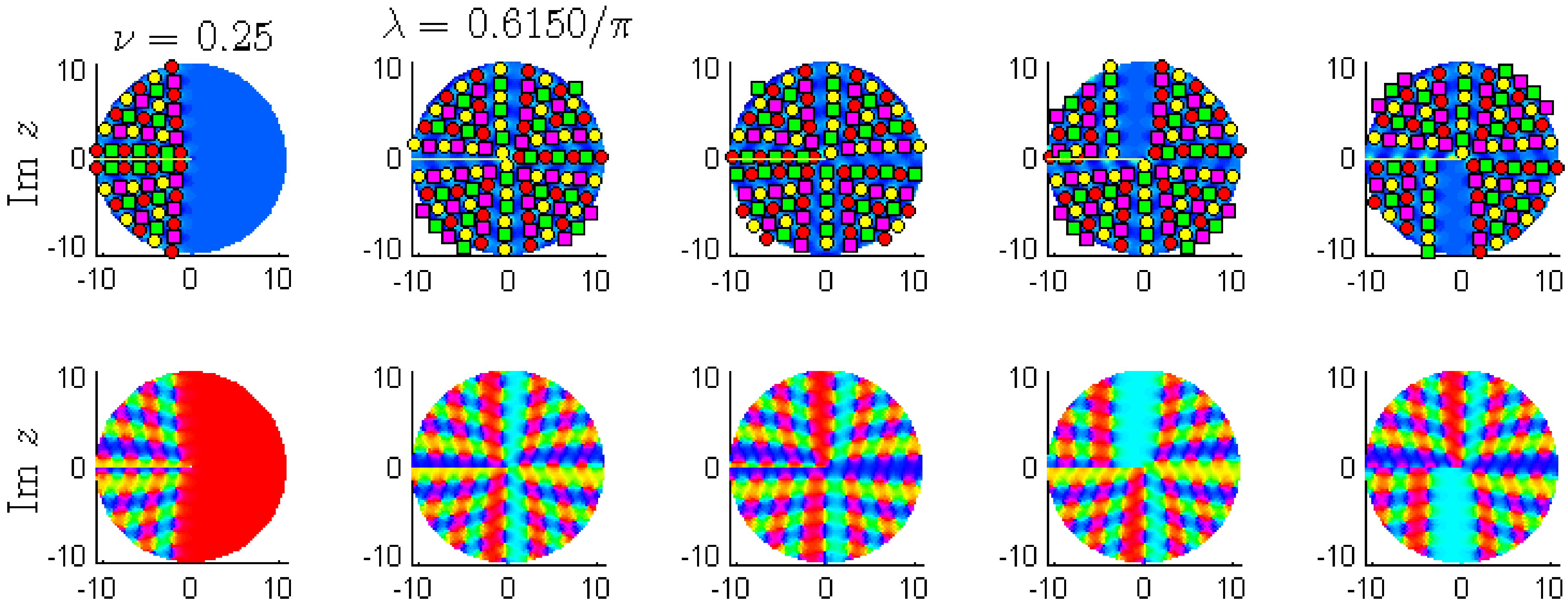

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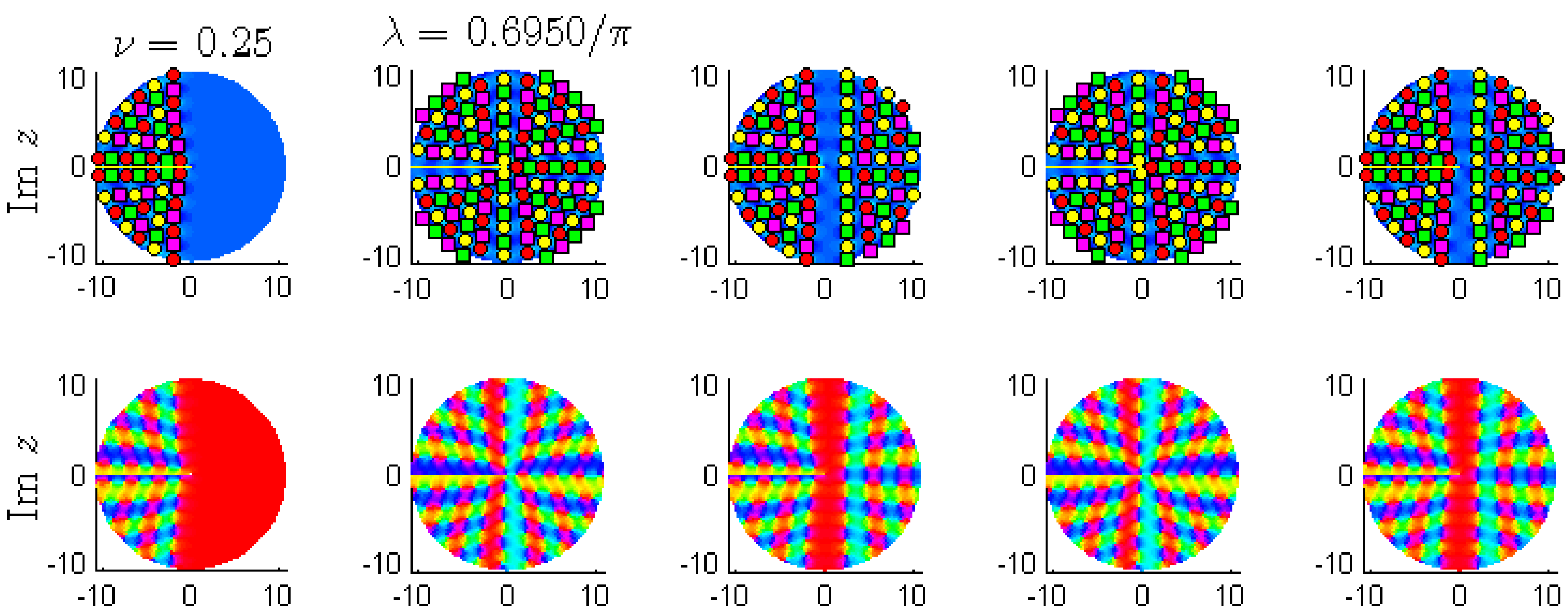

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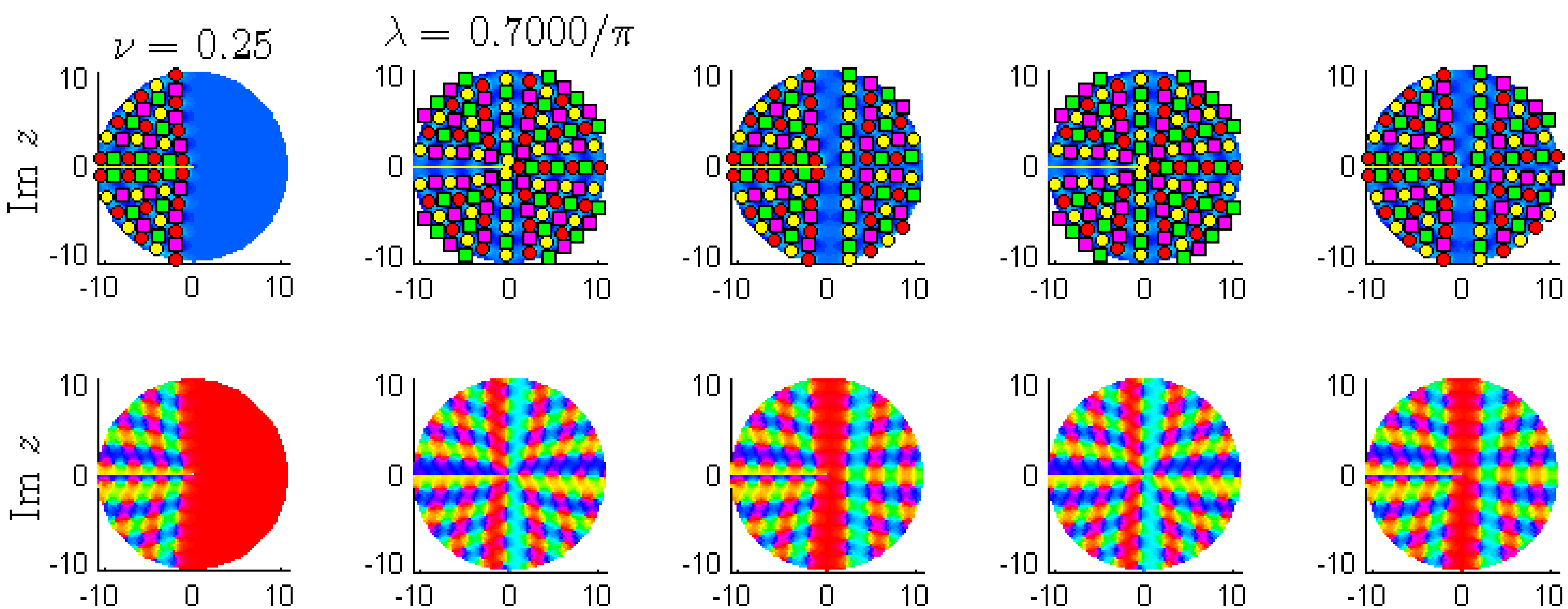

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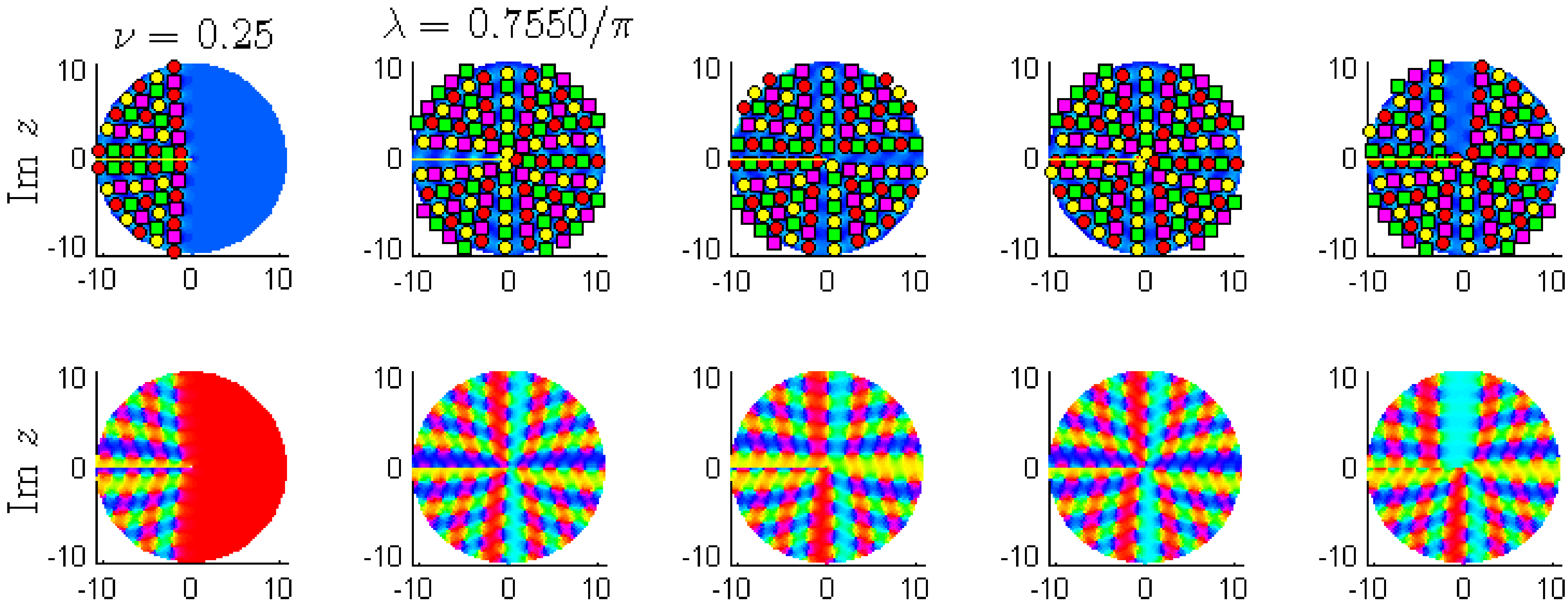

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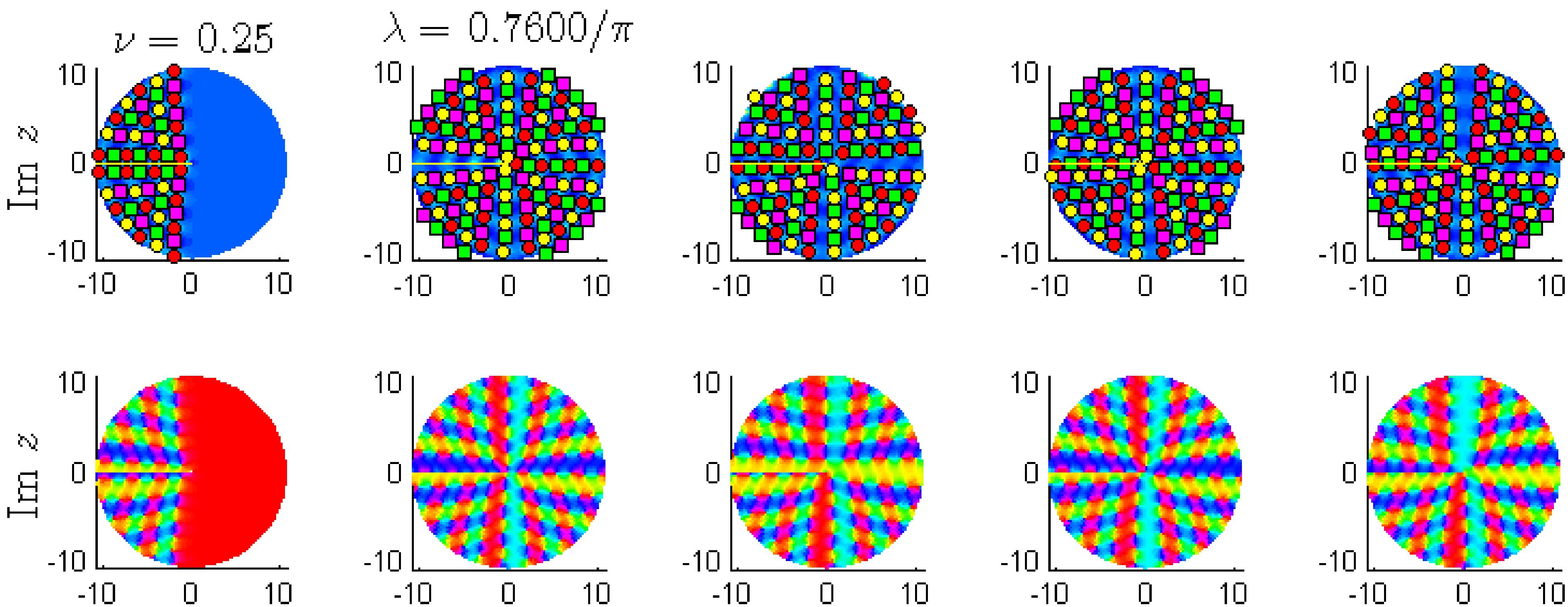

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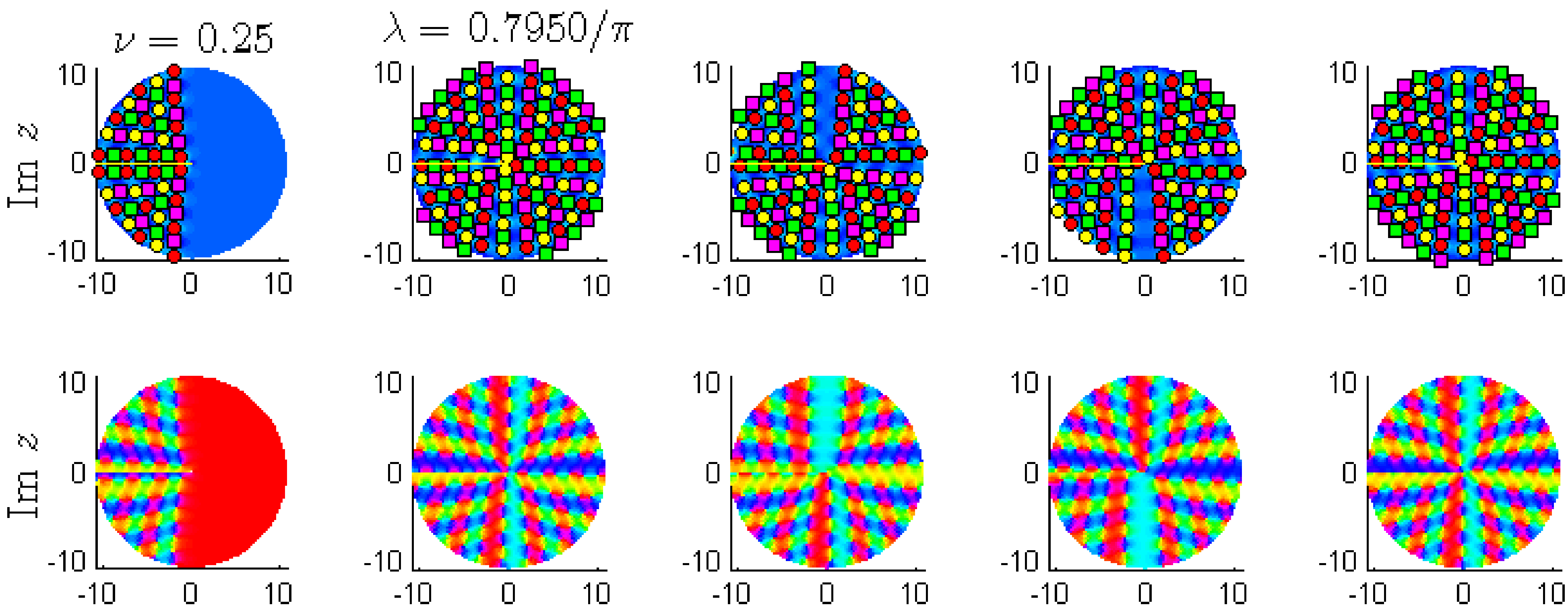

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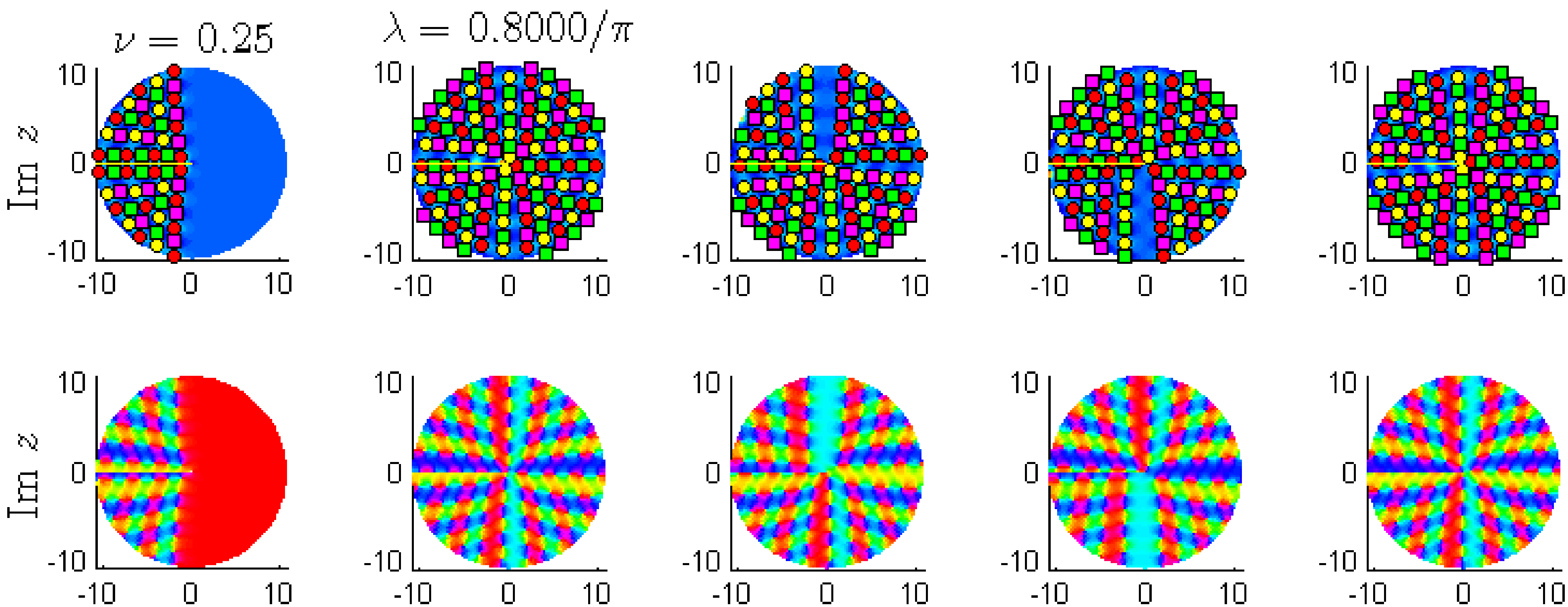

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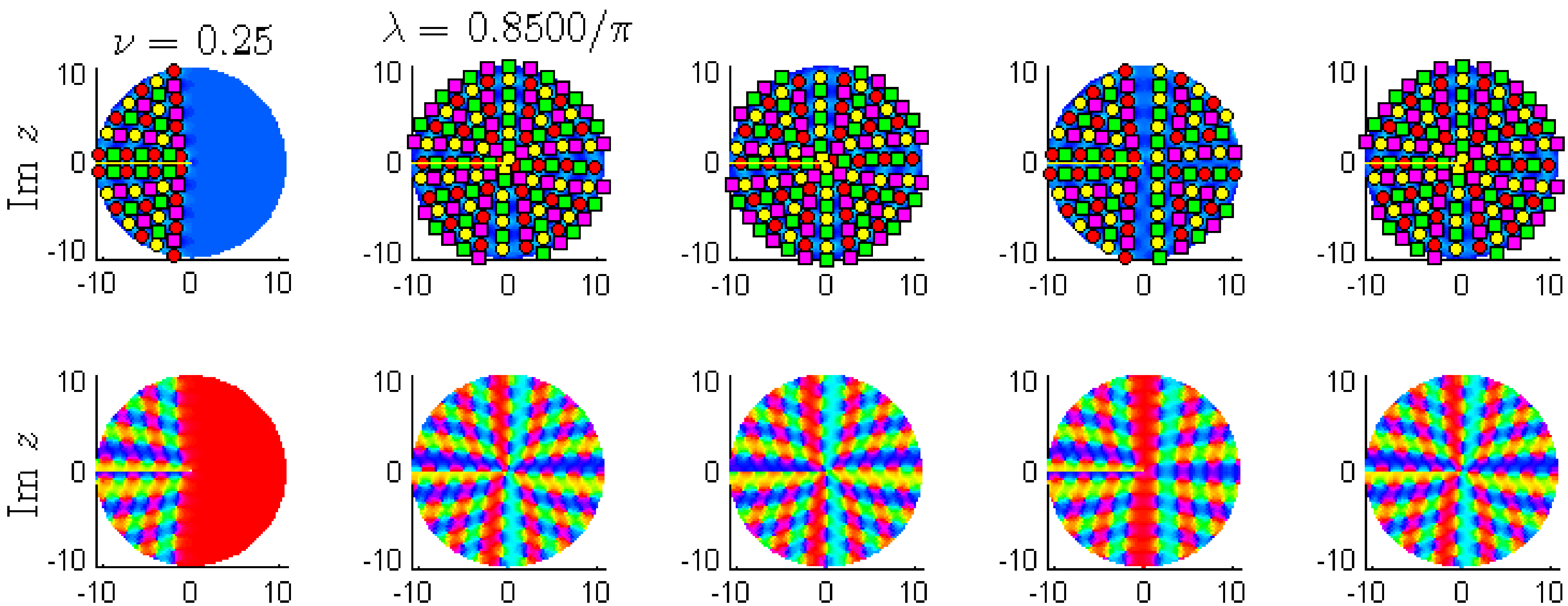

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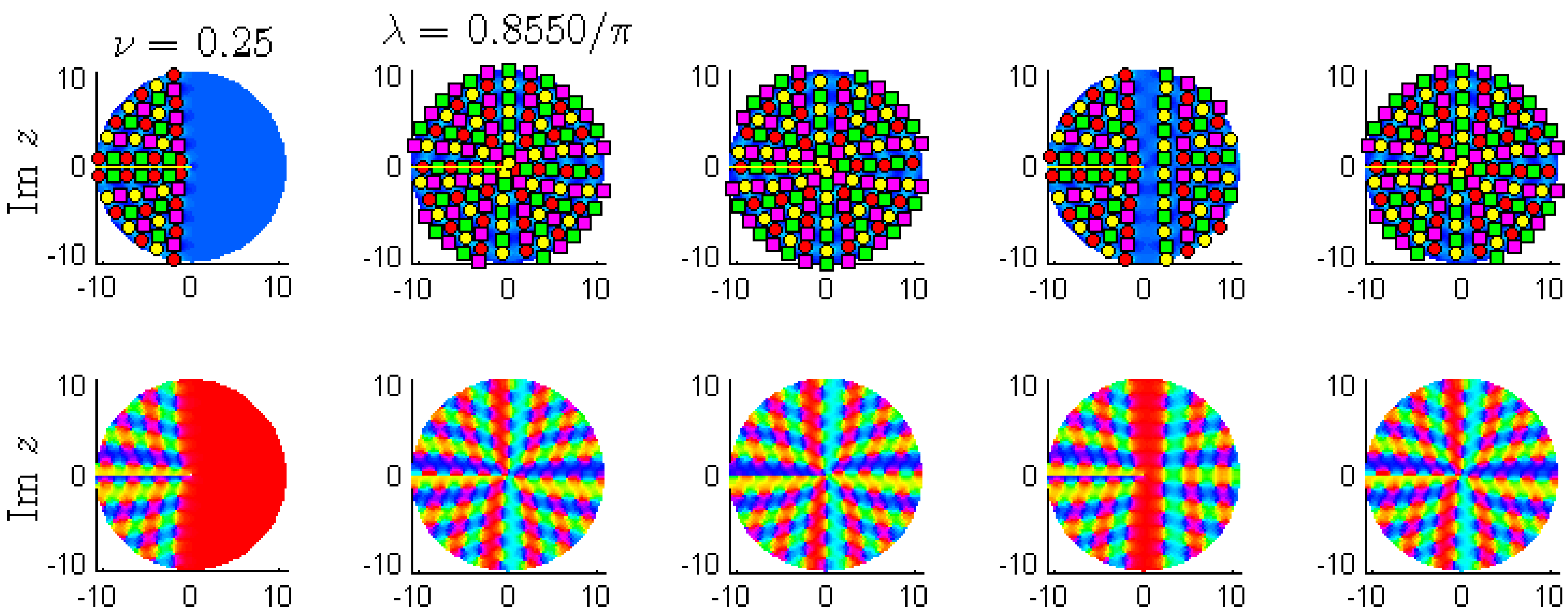

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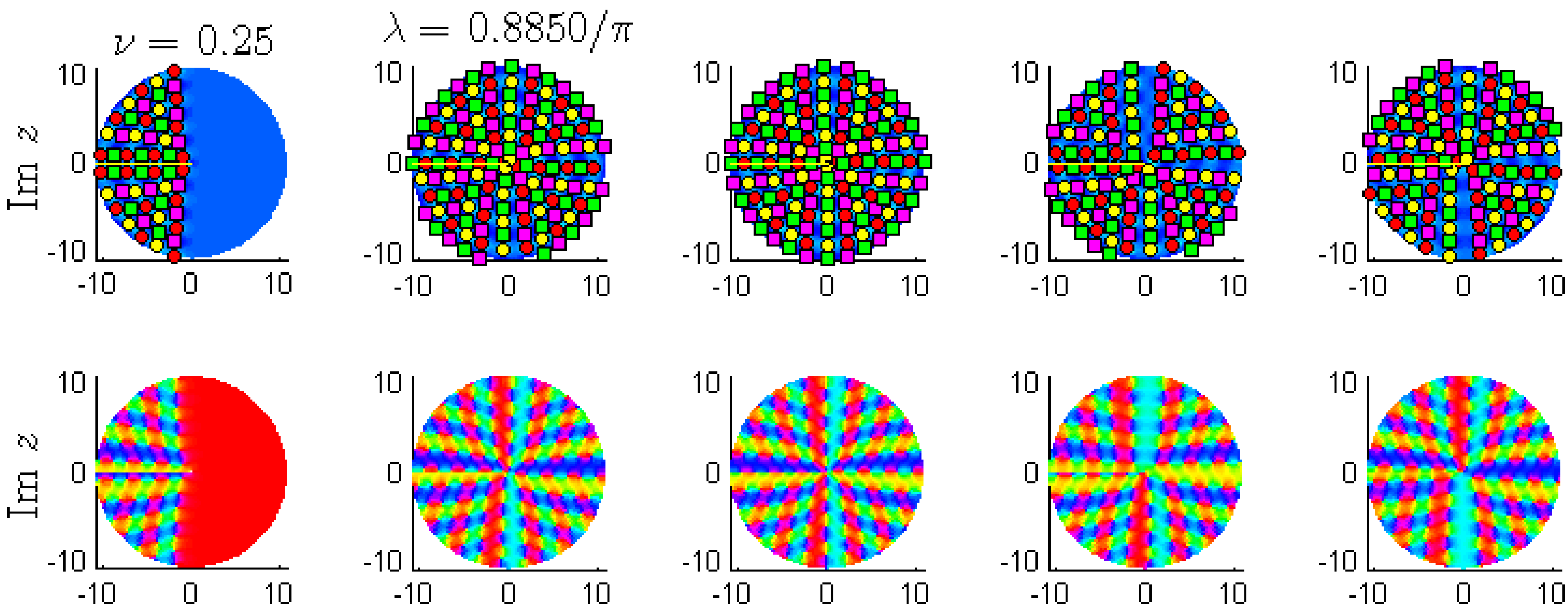

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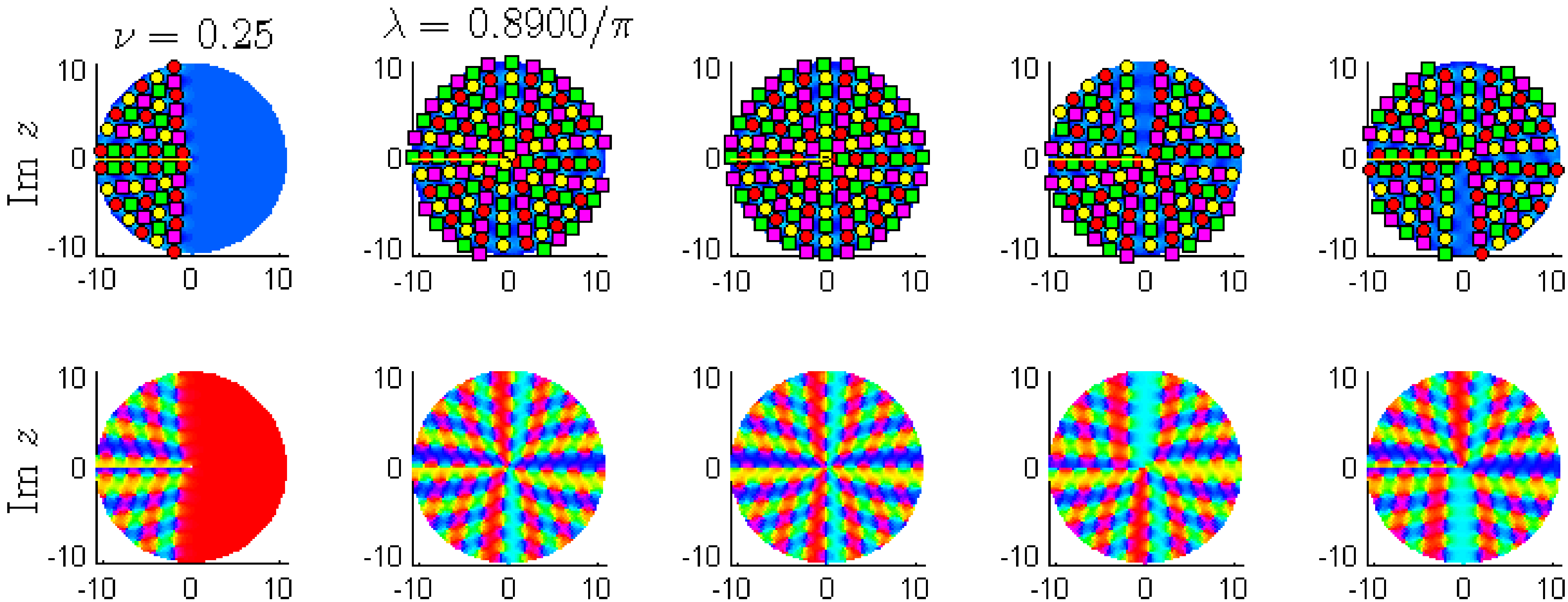

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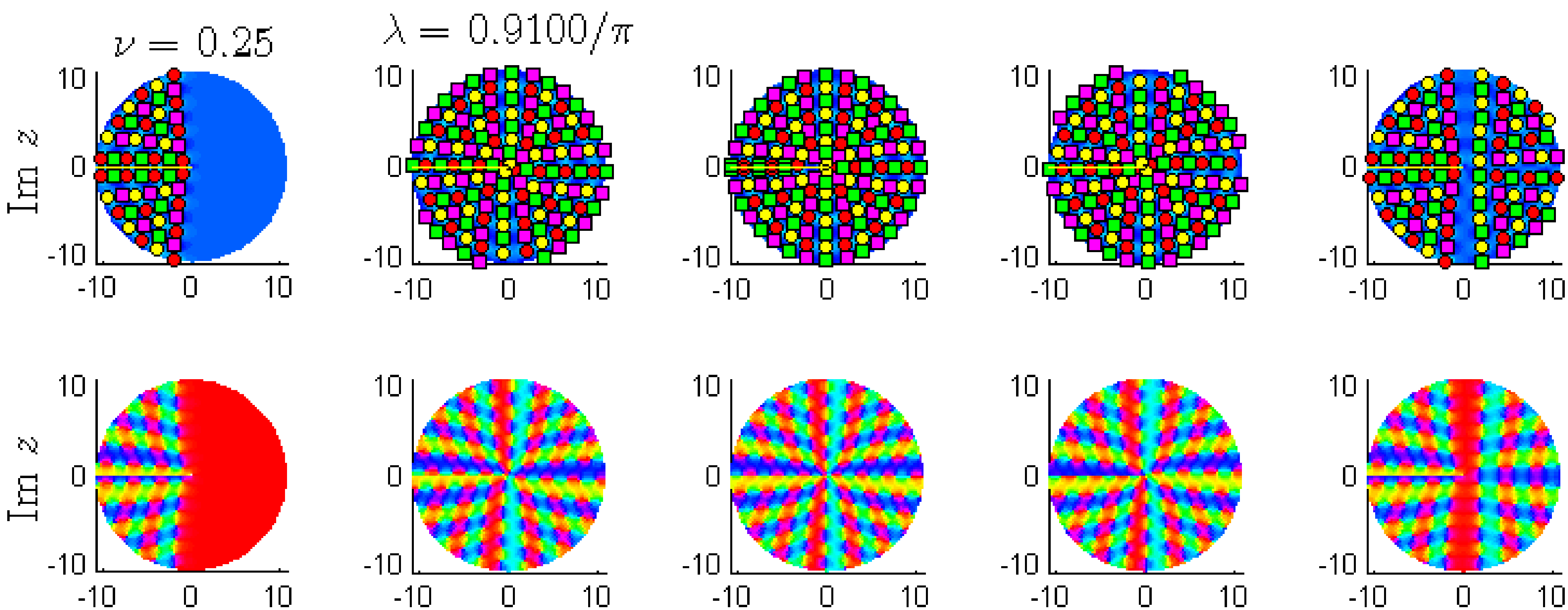

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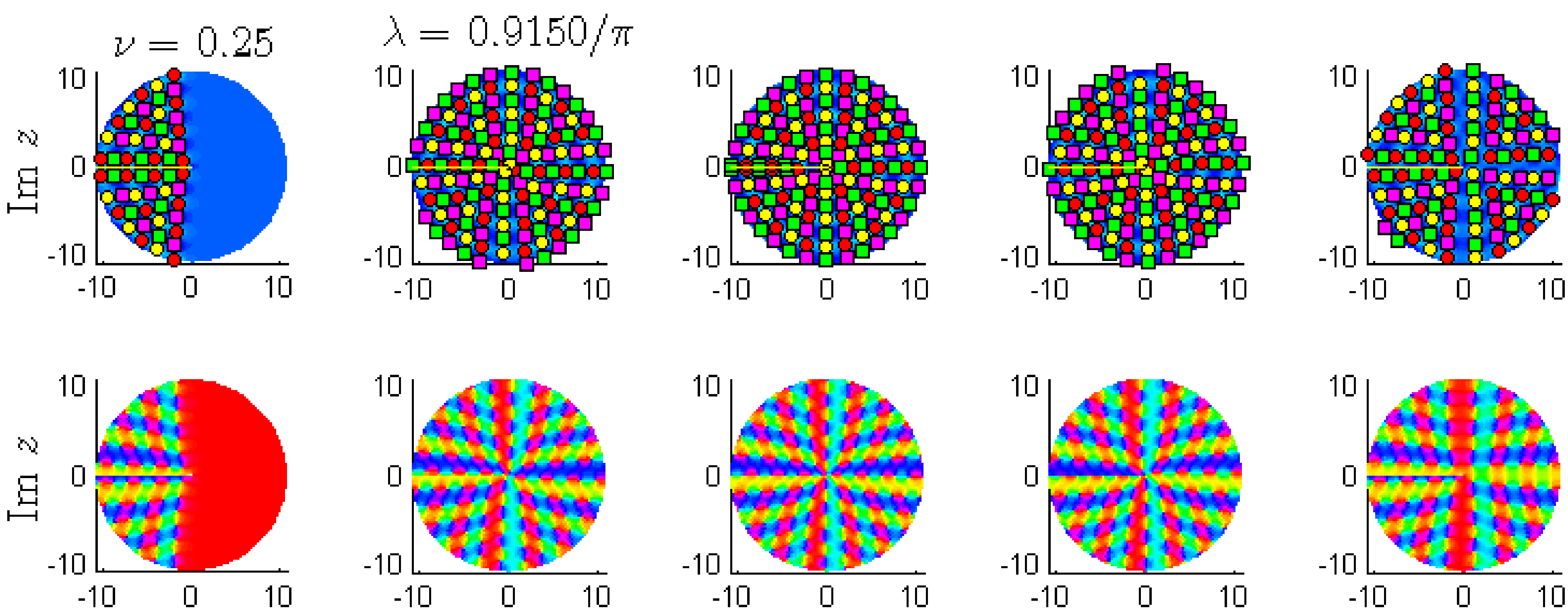

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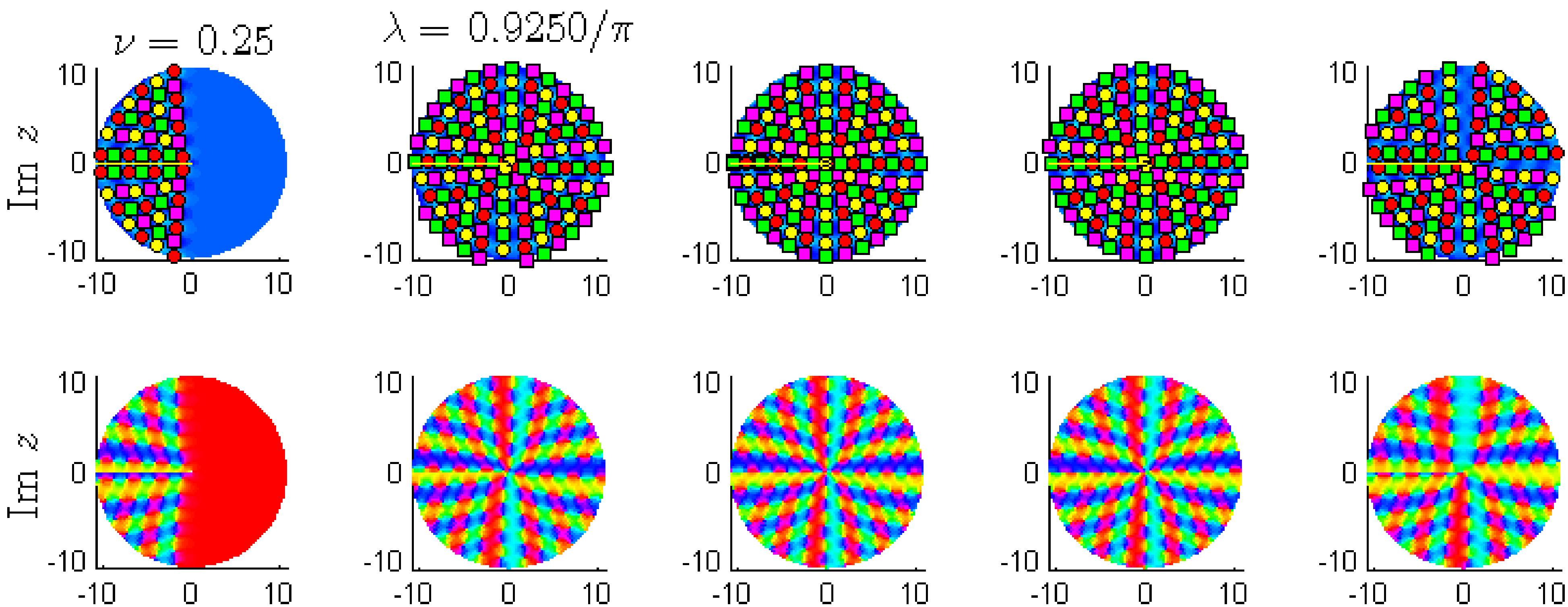

Tronquée P_{III} solutions

As $\boldsymbol{\lambda}$ is varied, multiple tronquée solutions occur on sheets other than the main sheet of the **MTW solutions, $u(z)$**

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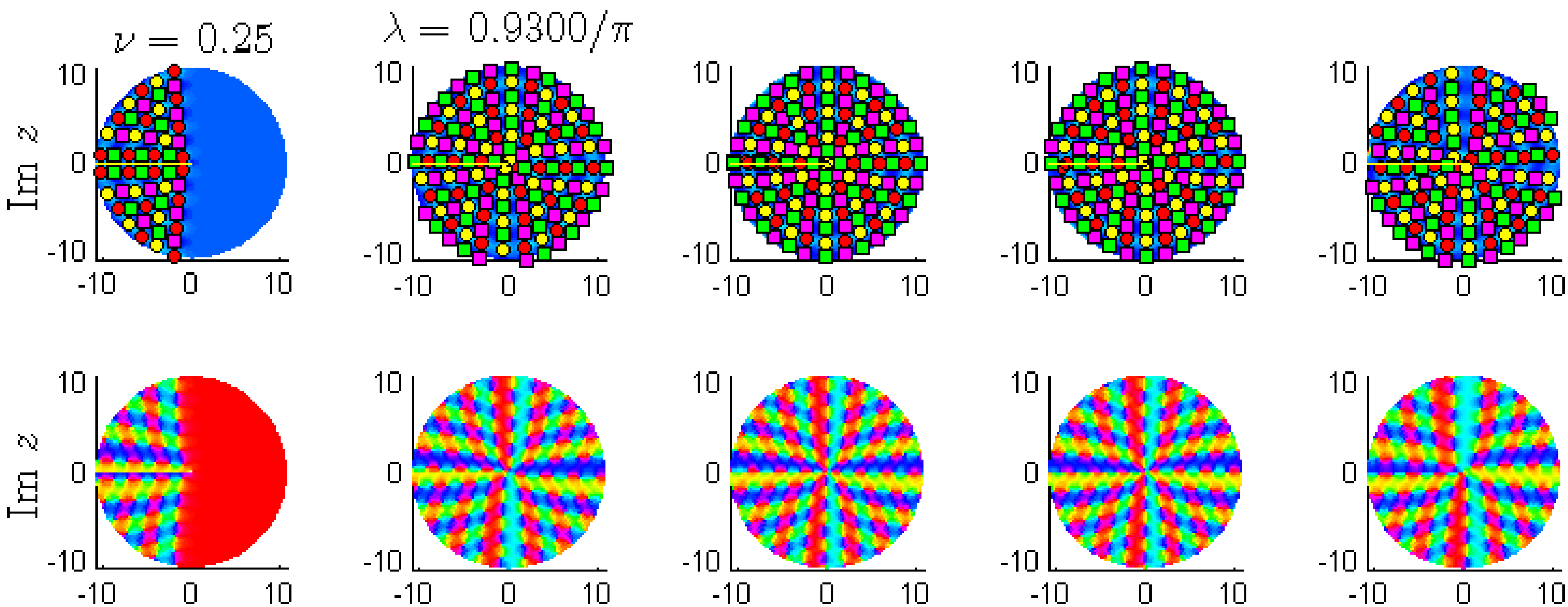

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$$u(z) \sim -1, \quad |z| \to \infty, \quad \left(k + \frac{1}{2}\right)\pi < \arg z < \left(k + \frac{3}{2}\right)\pi$$

$$\begin{aligned} k &= 8, 6, 4, 2 \\ k &= 8, 6, 4 \\ k &= 8, 6 \\ k &= 8 \end{aligned}$$

$$u(z) \sim 1, \quad |z| \to \infty, \quad \left(k + \frac{1}{2}\right)\pi < \arg z < \left(k + \frac{3}{2}\right)\pi$$

$$\begin{aligned} k &= 7, 5, 3 \\ k &= 7, 5 \\ k &= 7 \end{aligned}$$

Conjecture: The sequences continue indefinitely

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- $\mathbf{72}$ tronquée solutions if $\boldsymbol{\nu} \in \mathbb{Z}$. They have the properties

$$u(z) \sim i, \quad |z| \to \infty, \quad k\pi < \arg z < (k+1)\,\pi$$

$$\begin{aligned} k &= 8,7,6,5,4,3,2,1 \\ k &= 8,7,6,5,4,3 \\ k &= 8,7,6,5 \\ k &= 8,7 \end{aligned}$$

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Future plans

- Survey tronquée $\boldsymbol{P_{III}}$ solutions with $\boldsymbol{\alpha} = 1, \boldsymbol{\gamma} = 0$ and $\boldsymbol{\delta} = -1$
- Survey single-valued $\boldsymbol{P_{III}}$ solutions
- Survey triply branched $\boldsymbol{P_{III}}$ solutions
- Extend computational method for $\boldsymbol{P_{III}}$ to

$$\boldsymbol{P_V} \quad : \; \frac{d^2u}{dz^2} = \left(\frac{1}{2u} + \frac{1}{u-1}\right)\left(\frac{du}{dz}\right)^2 - \frac{1}{z}\frac{du}{dz} + \frac{(u-1)^2}{z^2}\left(\boldsymbol{\alpha} u + \frac{\boldsymbol{\beta}}{u}\right) + \boldsymbol{\gamma}\frac{u}{z} + \boldsymbol{\delta}\frac{u(u+1)}{u-1}$$

and

$$\boldsymbol{P_{VI}} \quad : \; \frac{d^2u}{dz^2} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-z}\right)\left(\frac{du}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{u-z}\right)\left(\frac{du}{dz}\right)$$
$$+ \frac{u(u-1)(u-z)}{z^2(z-1)^2}\left(\boldsymbol{\alpha} + \boldsymbol{\beta}\frac{z}{u^2} + \boldsymbol{\gamma}\frac{z-1}{(u-1)^2} + \boldsymbol{\delta}\frac{z(z-1)}{(u-z)^2}\right)$$

How to compute P_{III}?

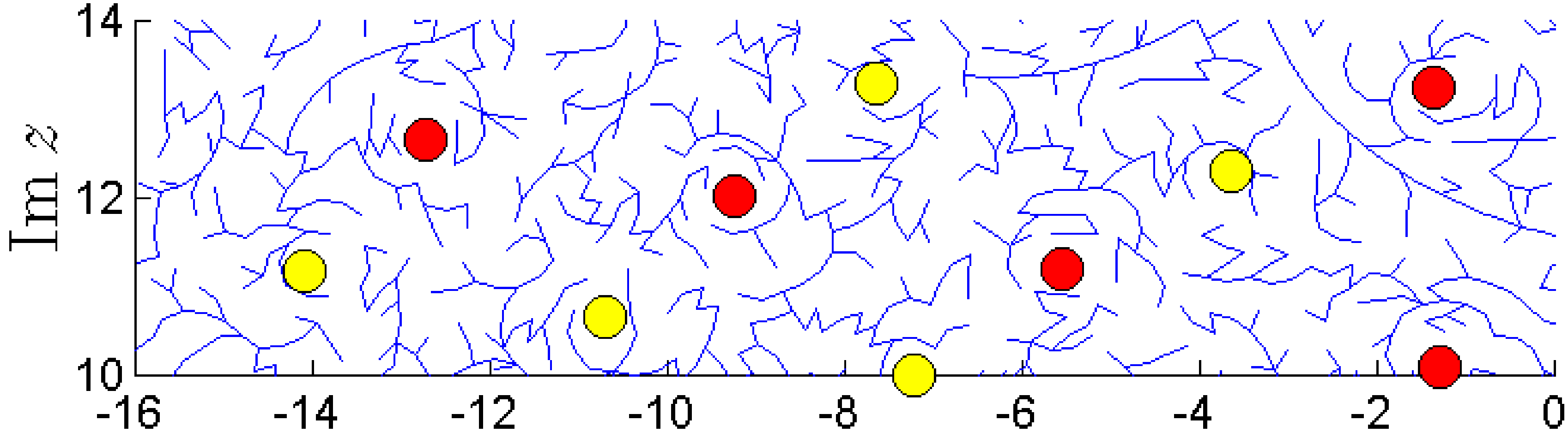
Challenges:

- **Moveable poles**: the 'Pole Field Solver' (PFS), **Fornberg & Weideman [2011]**

  $u(z), u'(z) \longrightarrow$ Taylor coefficients $\longrightarrow$ Padé approximation at $z + he^{i\theta}$

  Stage 1: pole avoidance on a coarse grid

  Stage 2: compute the solution on a fine grid



  Single-valued Painlevé transcendents: P_I, **F & W [2011]** ; P_{II}, **F & W [2014, 2015]** ; P_{IV}, **Reeger & F [2013, 2014]**

- **Multivaluedness**

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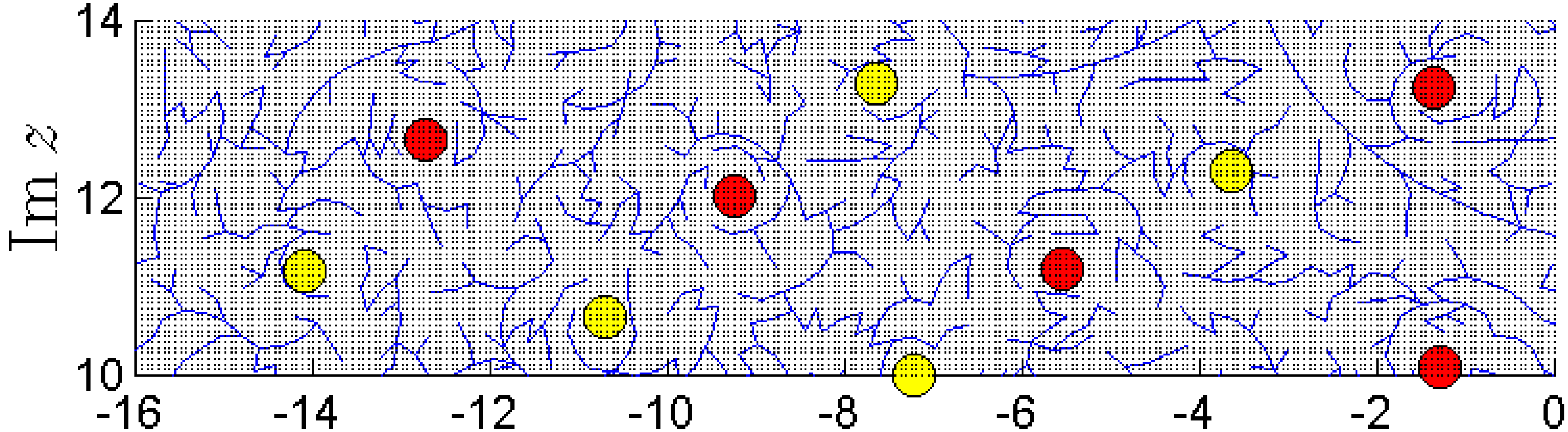

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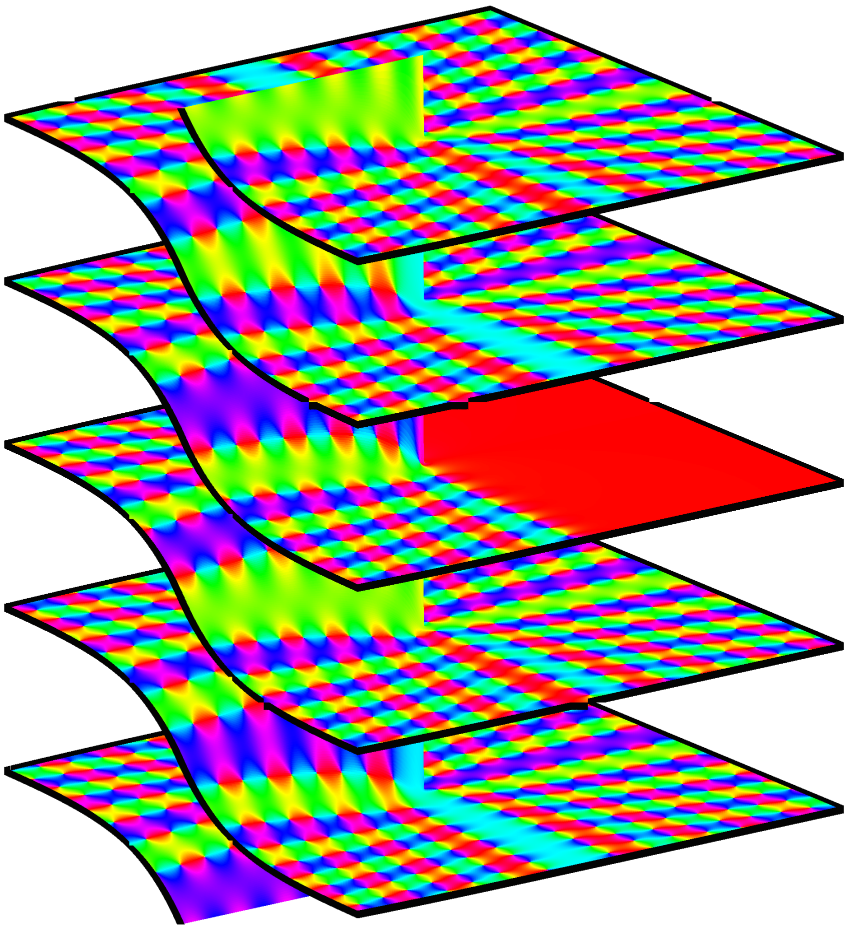
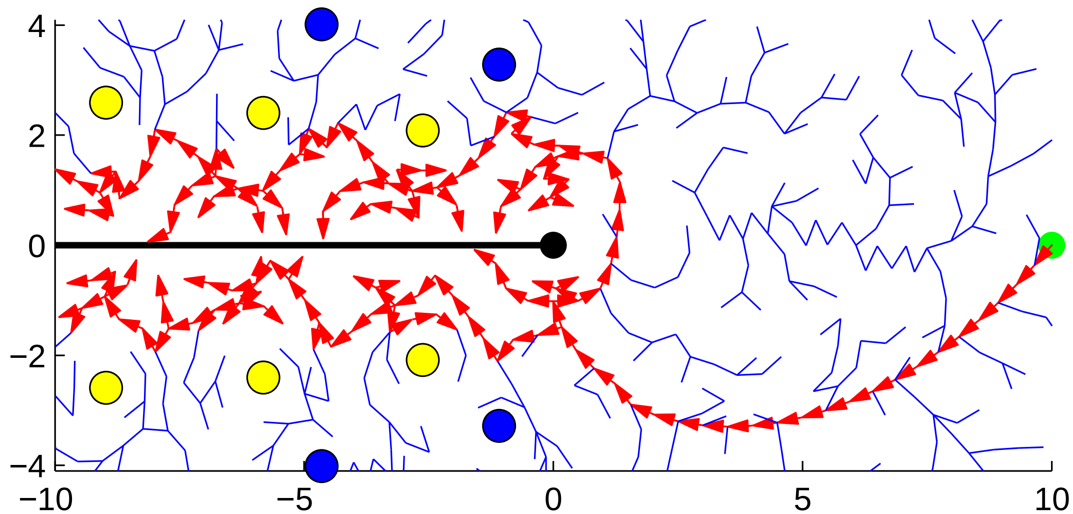
Second challenge: Multivaluedness

Make the paths run in the right direction





1st sheet

How to compute P_{III}?

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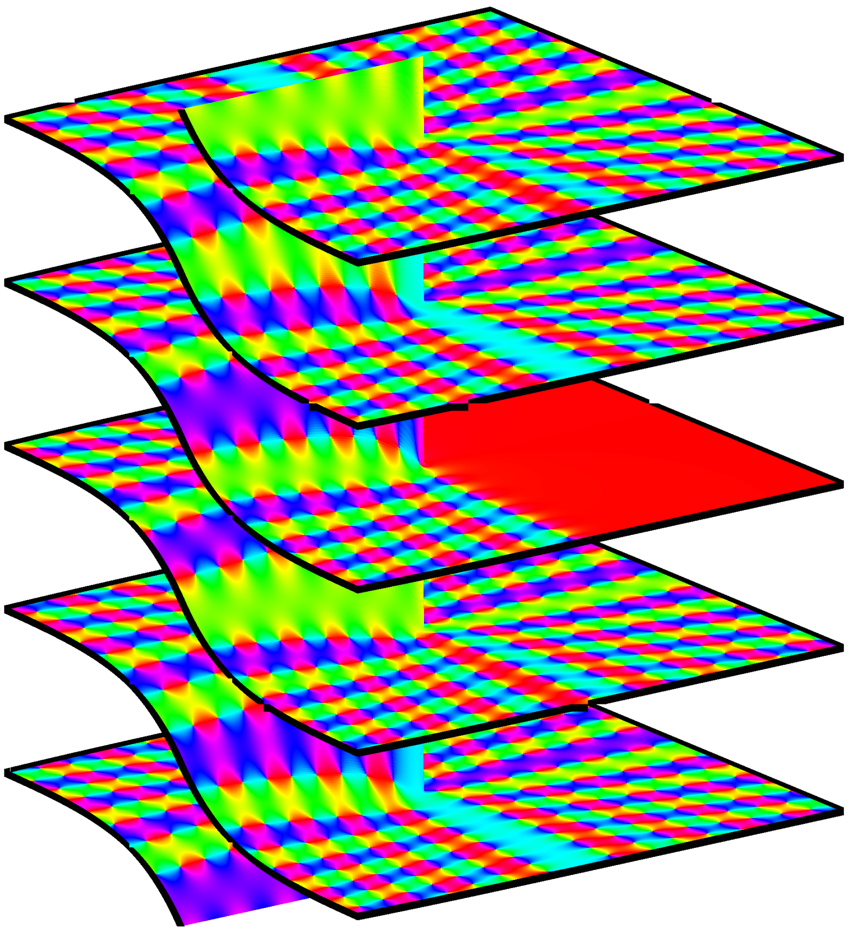
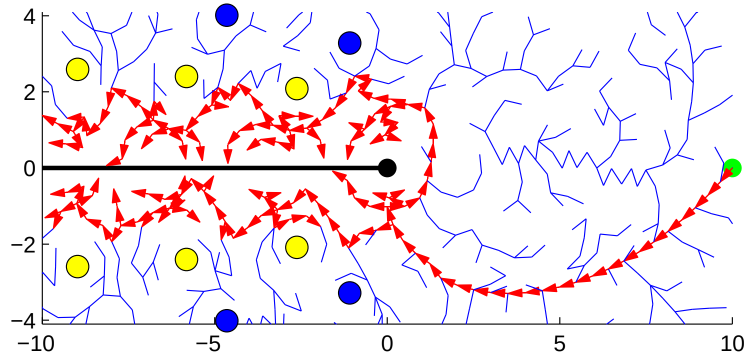
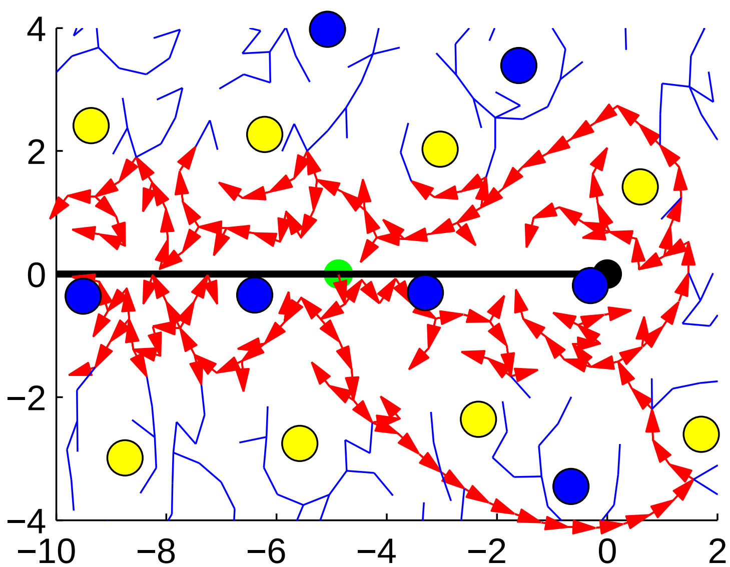




1st sheet



counterclockwise sheet

How to compute P_{III}?

Second challenge: Multivaluedness

$$P_{III}: \quad \frac{d^2u}{dz^2} = \frac{1}{u}\left(\frac{du}{dz}\right)^2 - \frac{1}{z}\frac{du}{dz} + \frac{\boldsymbol{\alpha} u^2 + \boldsymbol{\beta}}{z} + \boldsymbol{\gamma} u^3 + \frac{\boldsymbol{\delta}}{u}$$

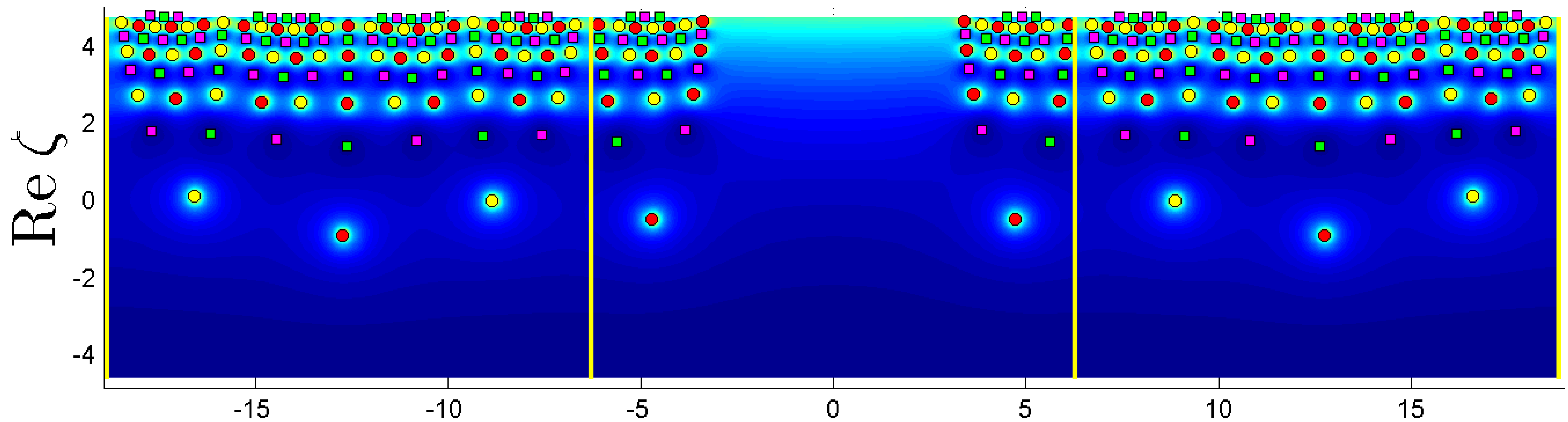
Let $z = e^{\zeta/2}$, $u(z) = e^{-\zeta/2} w(\zeta)$ in P_{III}, then

$$\widetilde{P_{III}}: \quad \frac{d^2w}{d\zeta^2} = \frac{1}{w}\left(\frac{dw}{d\zeta}\right)^2 + \frac{1}{4}\left(\boldsymbol{\alpha} w^2 + \boldsymbol{\gamma} w^3 + \boldsymbol{\beta} e^{\zeta} + \frac{\boldsymbol{\delta} e^{2\zeta}}{w}\right),$$

and all solutions of $\widetilde{P_{III}}$ are single-valued

Hinkkanen & Laine [2001]. ($z = 0 \Rightarrow \zeta = -\infty$)

How to compute P_{III}?



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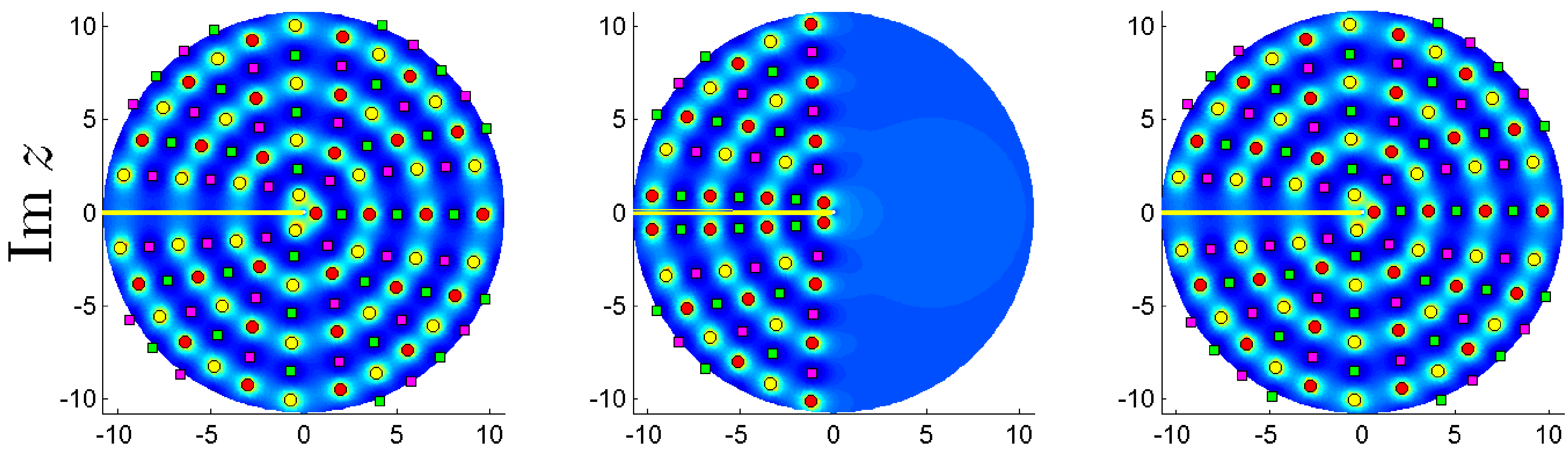
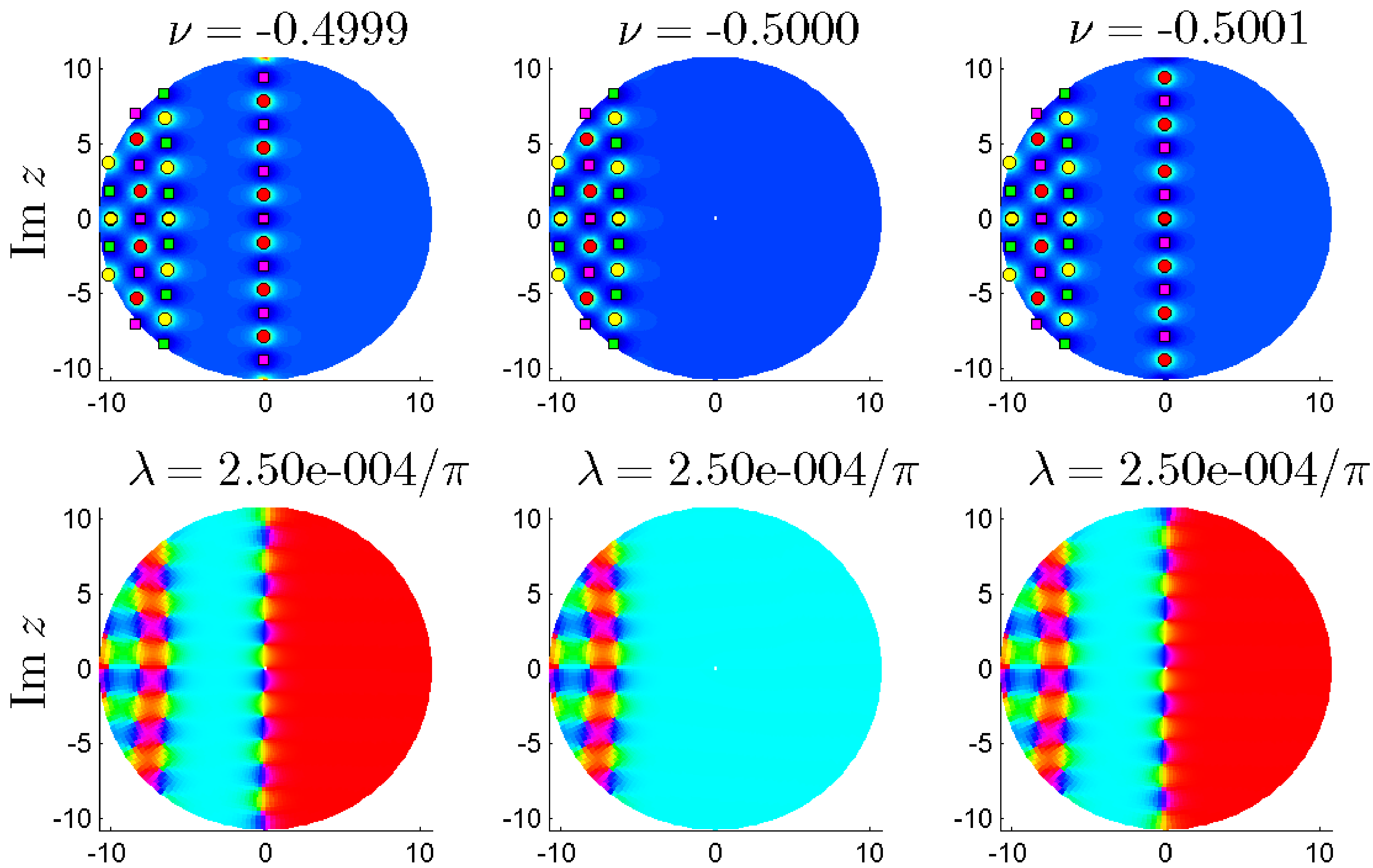


Tronquée P_{III} solutions

$$u(x) \sim 1 - \lambda\,\Gamma\!\left(\nu + \tfrac{1}{2}\right) 2^{-2\nu} x^{-\nu-(1/2)} e^{-2x}, \qquad x \to \infty$$

$$u(x) \sim B x^{\sigma}, \qquad x \to 0^{+}, \qquad -\frac{1}{\pi} < \lambda < \frac{1}{\pi}$$



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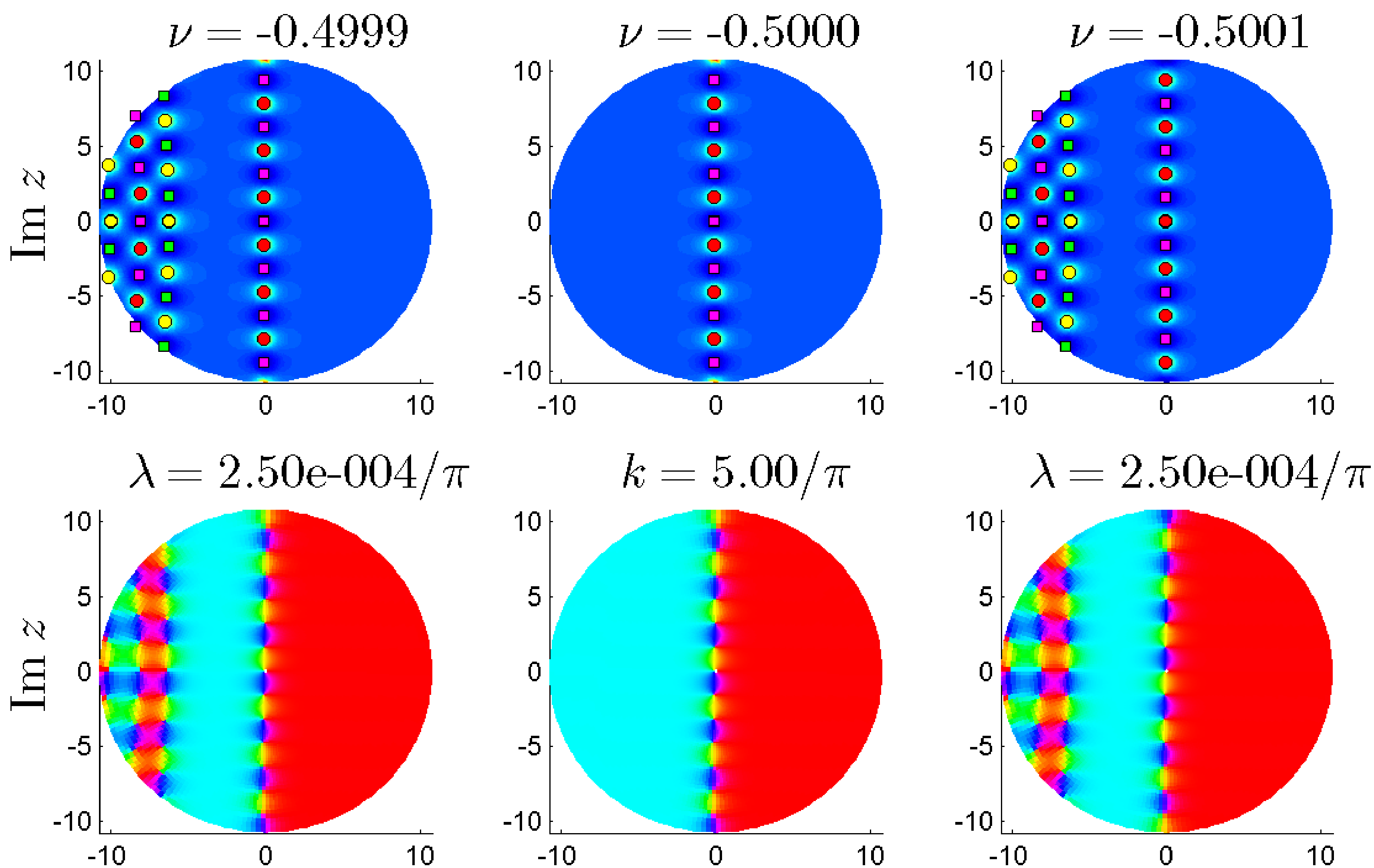

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