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A differential equation model for multi-class, multi-server queue networks with time dependent parameters.

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Zithulele Hospital

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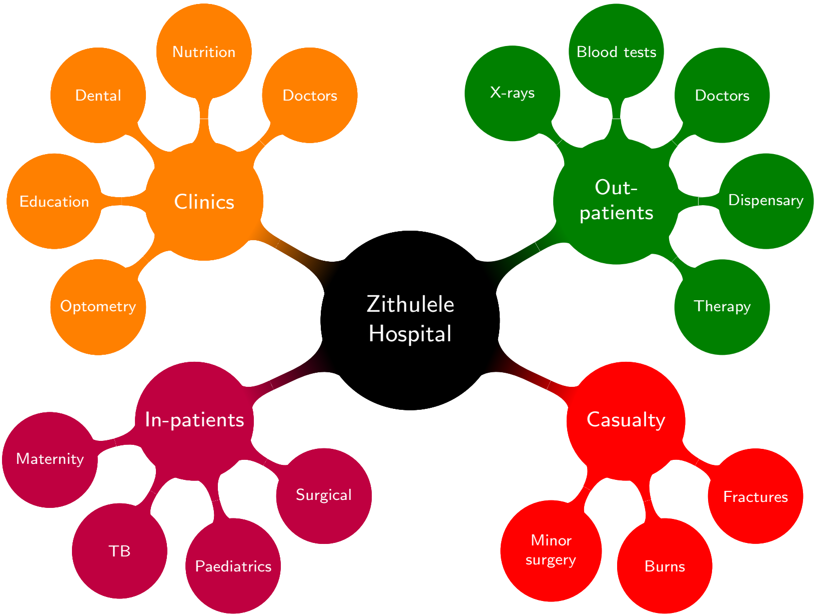
Hospital services



Aims

Minimise patient waiting times

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1. Understand the queueing process:

   Detailed model

   Data

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Minimise patient waiting times

1. Understand the queueing process:

   Detailed model

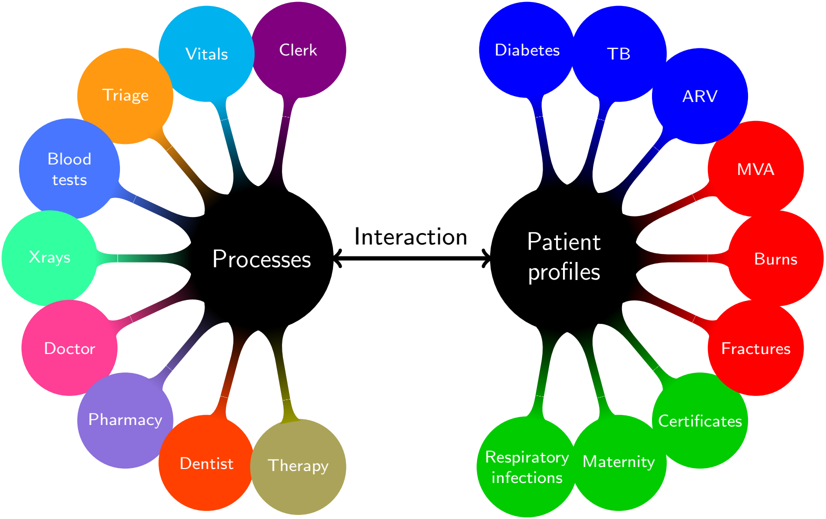
   Data

2. Make practical decisions:

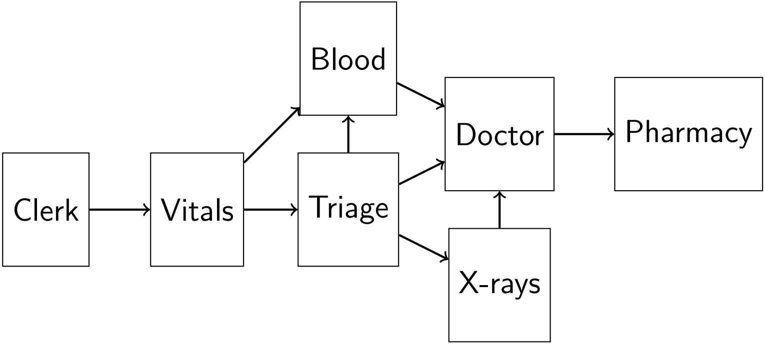
   Ongoing feedback

   Accessible with matric maths

General model

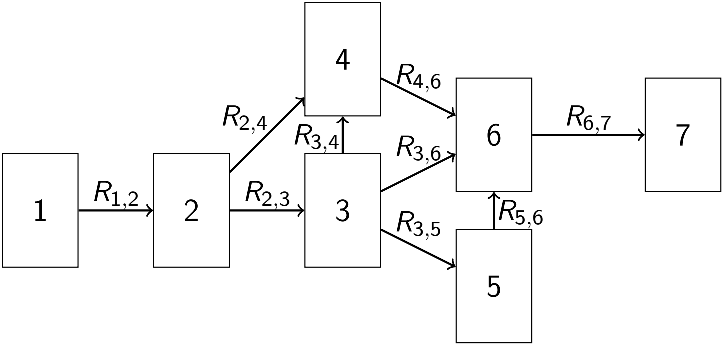


Queueing theory



Queueing theory

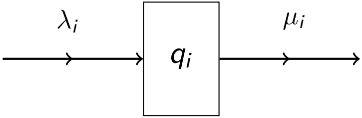
q_i: number of patients in the i^{th} queue
R: routing matrix



Queueing theory

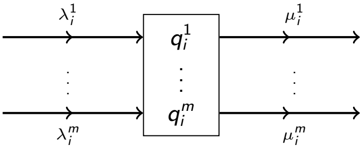
λ_i: arrival rate

μ_i: service rate



Patient profiles

λ_i^p: arrival rate for profile p
μ_i^p: service rate for profile p



Problems

- Multi-class queues (Kelly network)
- Non-stationary arrival rates $\lambda_i^p(t)$
- Time-dependent servers $s_i(t) \in \mathbb{N}_0$
- Transient queues - variation in traffic intensity
- Large state space

Fluid approximations

- Deterministic model for expected queue length
- First-order DE's
- Approximate discrete queues with continuous functions
- Represent arrivals/exits with continuous (mean) flows

Equations

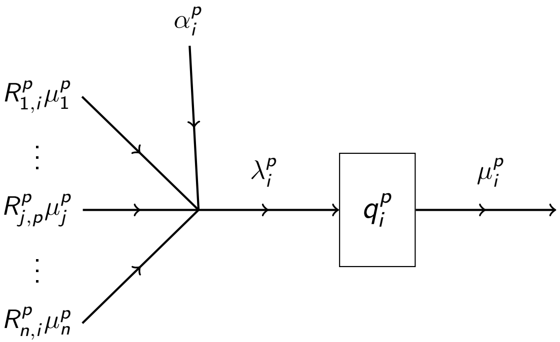
$$q_i(t) = \sum_{p=1}^{m} q_i^p(t)$$

$$\frac{\mathrm{d}q_i^p}{\mathrm{d}t} = \lambda_i^p(t) - \mu_i^p(t)$$

Arrival rate

$$\lambda_i^p(t) = \alpha_i^p(t) + \sum_{j=1}^{n} R_{j,i}^p \mu_j^p(t)$$

α_i^p: external arrival rate for profile p
$R_{j,i}^p$: probability of moving from process $j \rightarrow i$



Traffic intensity

$$\rho_i(t) = \frac{\sum_{p=1}^{m} \lambda_i^p(t)\tau_i^p}{s_i(t)}$$

τ_i^p: minutes to treat patient type p at process i
$s_i(t)$: staff on duty at process i

Case 1: no backlog

- $\rho_i(t) \leq 1$
  and
- $\sum_{p=1}^{m} q_i^p(t) = 0$

Case 2: backlog

- $\rho_i(t) > 1$
  or
- $\sum_{p=1}^{m} q_i^p(t) > 0$

State function

$$\phi_i(t) = \begin{cases} 1 & \sum_{p=1}^{m} \lambda_i^p(t)\tau_i^p \leq s_i(t) \quad \&\& \quad q_i(t) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Service rate: no backlog

Patients are treated on arrival:

$$\mu_i^p(t) = \lambda_i^p(t)$$

Service rate: backlog

New arrivals join the queue.
Staff must divide their time between different patient profiles:

$$\mu_i^p(t) = \frac{\beta_i^p(t) s_i(t)}{\tau_i^p}$$

Service rate: backlog

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Weights β_i^p are proportional to the number of patients and their treatment needs:

$$\beta_i^p(t) = c_i(t) q_i^p(t) \tau_i^p$$

Service rate: backlog

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Weights β_i^p are proportional to the number of patients and their treatment needs:

$$\beta_i^p(t) = c_i(t)q_i^p(t)\tau_i^p$$

$$c_i(t) = \frac{1}{\sum_{p=1}^{m} q_i^p(t)\tau_i^p}$$

Service rate

$$\mu_i^p(t) = \phi_i \lambda_i^p(t) + (1 - \phi_i) \frac{s_i(t) q_i^p(t)}{\sum_{k=1}^m q_i^k(t) \tau_i^k + \phi_i}$$

$$\phi_i(t) = \begin{cases} 1 & \sum_{p=1}^m \lambda_i^p(t) \tau_i^p \leq s_i(t) \quad \&\& \quad q_i(t) = 0 \\ 0 & \textit{otherwise}. \end{cases}$$

Initial DE model

$$\frac{\mathrm{d}q_i^p}{\mathrm{d}t} = \lambda_i^p(t) - \mu_i^p(t)$$

$$\lambda_i^p(t) = \alpha_i^p(t) + \textstyle\sum_{j=1}^{n} R_{j,i}^p \mu_j^p(t)$$

$$\mu_i^p(t) = \phi_i \lambda_i^p(t) + (1-\phi_i)\frac{s_i(t) q_i^p(t)}{\sum_{k=1}^{m} q_i^k(t)\tau_i^k + \phi_i}$$

$$\phi_i(t) = \begin{cases} 1 & \sum_{p=1}^{m} \lambda_i^p(t)\tau_i^p \leq s_i(t) \quad \&\& \quad q_i(t) = 0 \\ 0 & otherwise. \end{cases}$$

Initial DE model

Unknown functions: q_i^p, λ_i^p, μ_i^p, ϕ_i

Function	Units	Equation
q_i^p	Patients	Differential
λ_i^p	Patients/time	Algebraic
μ_i^p	Patients/time	Algebraic
ϕ_i	Binary	Algebraic

Substitutions

$$\lambda_i^p(t) = \frac{\mathrm{d}\Lambda_i^p}{\mathrm{d}t} \qquad \Lambda_i^p(t) = \int_0^t \lambda_i^p(x)dx$$

$$\mu_i^p(t) = \frac{\mathrm{d}U_i^p}{\mathrm{d}t} \qquad U_i^p(t) = \int_0^t \mu_i^p(x)dx$$

Substitutions

$$\frac{\mathrm{d}q_i^p}{\mathrm{d}t} = \frac{\mathrm{d}\Lambda_i^p}{\mathrm{d}t} - \frac{\mathrm{d}U_i^p}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\Lambda_i^p}{\mathrm{d}t} = \alpha_i^p(t) + \sum_{j=1}^n R_{j,i}^p \frac{\mathrm{d}U_i^p}{\mathrm{d}t}$$

$$\frac{\mathrm{d}U_i^p}{\mathrm{d}t} = \phi_i \left(\alpha_i^p(t) + \sum_{j=1}^n R_{j,i}^p \frac{\mathrm{d}U_j^p}{\mathrm{d}t} \right) + (1-\phi_i) \frac{s_i(t) q_i^p(t)}{\sum_{k=1}^m q_i^k(t)\tau_i^k + \phi_i}$$

$$\phi_i(t) = \begin{cases} 1 & \sum_{k=p}^m \frac{\mathrm{d}\Lambda_i^p}{\mathrm{d}t}\tau_i^p \le s_i(t) \quad \&\& \quad q_i(t) = 0 \\ 0 & \textit{otherwise}. \end{cases}$$

Substitutions

$$q_i^p(t) = \Lambda_i^p(t) - U_i^p(t)$$

$$\Lambda_i^p(t) = A_i^p(t) + \sum_{j=1}^{n} R_{j,i}^p U_i^p(t)$$

Substitutions

$$\frac{\mathrm{d}U_i^p}{\mathrm{d}t} = \phi_i \left(\alpha_i^p(t) + \sum_{j=1}^{n} R_{j,i}^p \frac{\mathrm{d}U_j^p}{\mathrm{d}t} \right) + (1-\phi_i) \frac{s_i(t) q_i^p(t)}{\sum_{k=1}^{m} q_i^k(t)\tau_i^k + \phi_i}$$

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$$q_i^p(t) \quad \rightarrow \quad A_i^p(t) + \sum_{j=1}^n R_{j,i}^p U_i^p(t) - U_i^p(t)$$

$$\frac{\mathrm{d}\Lambda_i^p}{\mathrm{d}t} \quad \rightarrow \quad \alpha_i^p(t) + \sum_{j=1}^n R_{j,i}^p \frac{\mathrm{d}U_i^p}{\mathrm{d}t}$$

Final DE model

$$\frac{\mathrm{d}U_i^p}{\mathrm{d}t} = \phi_i \left(\alpha_i^p(t) + \sum_{j=1}^{n} R_{j,i}^p \frac{\mathrm{d}U_j^p}{\mathrm{d}t} \right) + (1-\phi_i) \left(\frac{s_i(t) \left(A_i^p(t) + \sum_{j=1}^{n} R_{j,i}^p U_j^p - U_i^p \right)}{\sum_{k=1}^{m} \tau_i^k \left(A_i^k(t) + \sum_{j=1}^{n} R_{j,i}^k U_j^k - U_i^k \right) + \phi_i} \right)$$

$$\phi_i(t) = \begin{cases} 1 & \sum_{p=1}^{m} (\alpha_i^p(t) + \sum_{j=1}^{n} R_{j,i}^p \frac{\mathrm{d}U_i^p}{\mathrm{d}t}) \tau_i^p \leq s_i(t) \\ & \&\& \quad A_i^p(t) + \sum_{j=1}^{n} R_{j,i}^p U_j^p - U_i^p = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Numerical solution

1. Solve DE's for U_i^p and ϕ_i:
   Initialise $t = 0$, $U_i^p(0) = 0$ and $\phi_i(0) = 0$.

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   1. Calculate $U_i^p(t + \Delta t)$
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    2. Update $\phi_i(t + \Delta t)$
    3. If $\phi_i(t + \Delta t) = \phi_i(t)$:
        - $t = t + \Delta t$

Numerical solution

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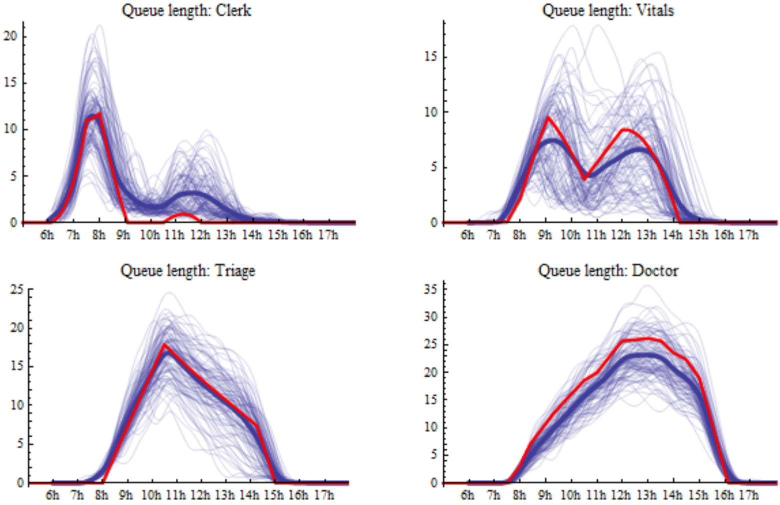
   1. Calculate $U_i^p(t + \Delta t)$
   2. Update $\phi_i(t + \Delta t)$
   3. If $\phi_i(t + \Delta t) = \phi_i(t)$:
      - $t = t + \Delta t$
   4. If $\phi_i(t + \Delta t) \neq \phi_i(t)$:
      - Locate discontinuity at $t + \delta$
      - Calculate $U_i^p(t + \delta)$
      - Calculate $\phi_i(t + \delta^+)$
      - $t = t + \delta$

Results: Queue length



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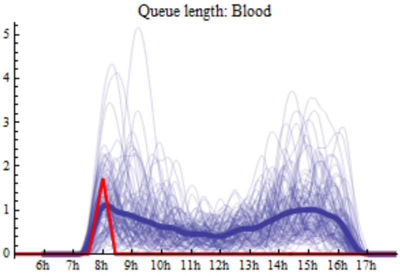
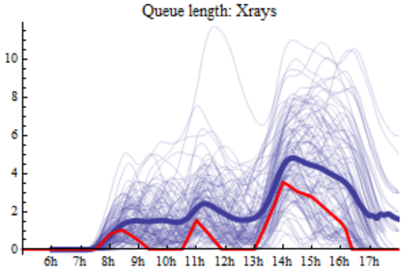
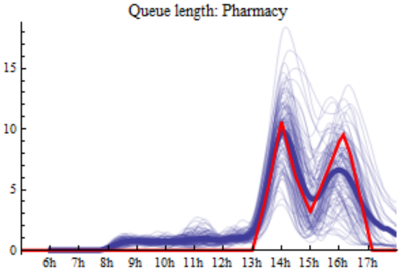
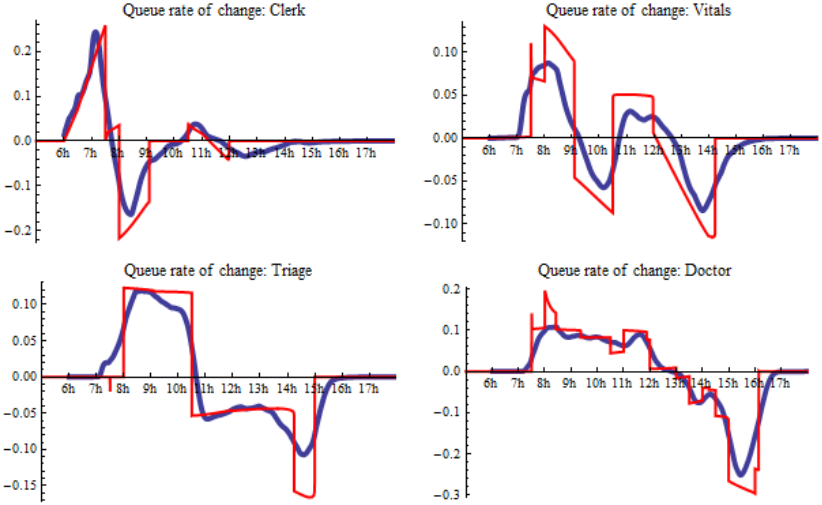


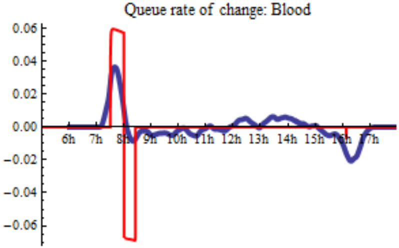


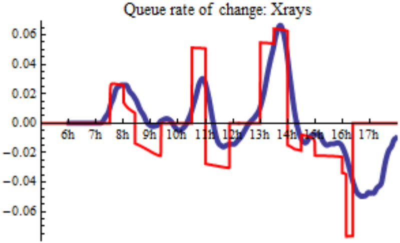


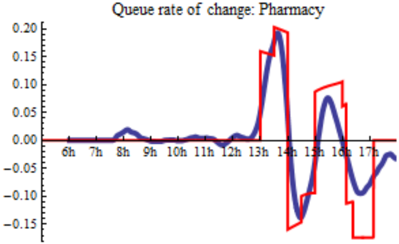
Results: Rate of change



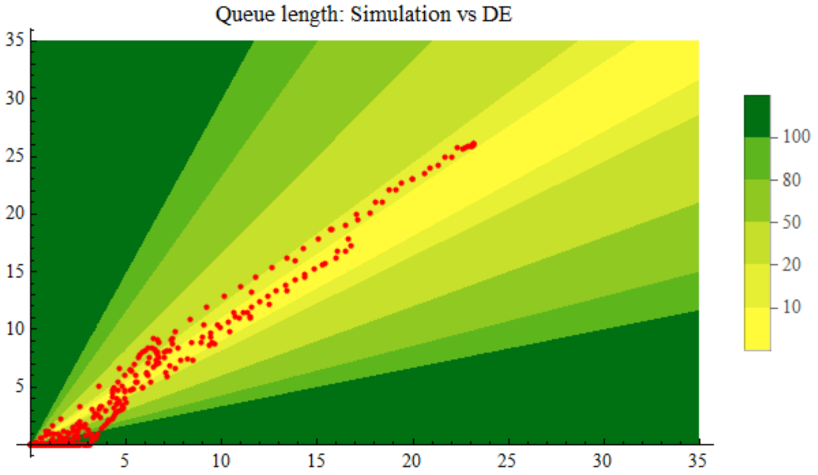
Results: Rate of change



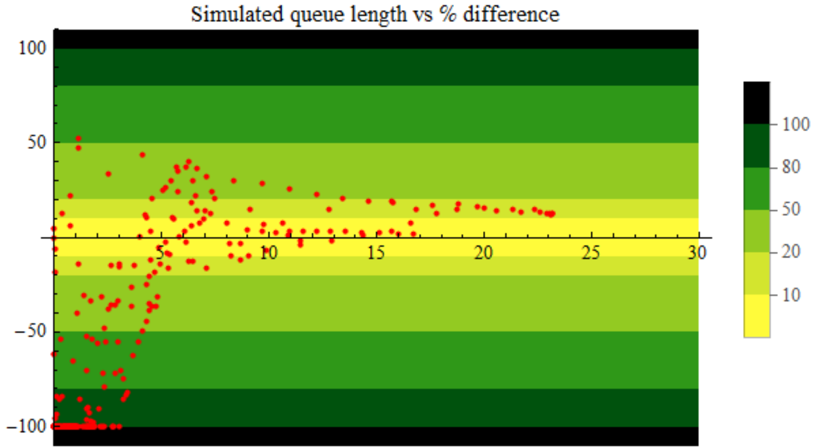




Results: Accuracy



Results: Accuracy



Conclusion

- DE results give less information than simulations
- Fairly accurate for long queues/high traffic intensity
- Can usually predict queue growth

References

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