An ultraspherical spectral method for fractional integral and differential equations of half-integer order

SANUM 2016
Stellenbosch University
22nd March 2016

Nick Hale
Stellenbosch University
(Joint work with Sheehan Olver @ USYD)

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ULTRASPHERICAL SPECTRAL (US) METHOD

Ultraspherical spectral method
(A very quick recap)

- Olver & Townsend, "A fast and well-conditioned spectral method" (2012)
- Two key ingredients: (banded differentiation & conversion operators)

$$\frac{d}{dx}T_n(x) = nU_{n-1}(x), \qquad T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x))$$

- Natural factorization of the Chebyshev (coefficient) differentiation matrix:

$$D_{tau} = S^{-1}D_{US}$$

- When solving ODEs:

$$u' + u = f \longrightarrow (D_{US} + S)\mathbf{u} = S\mathbf{f}$$

- Higher-order version uses similar relationships for Ultraspherical polynomials
- Advantages:
  - Fast (banded matrices)
  - Well-conditioned (ill-conditioning is in S^{-1})

Ultraspherical spectral method

(A very quick recap)

$D_{tau} = S^{-1} D_{US}$:

```
[ 0     1           3           5 ]
[             4           8       ]
[                   6          10 ] =
[                         8       ]
[                              10 ]
[                                 ]

[ 1         -0.5                  ]^-1    [ 0     1                         ]
[      0.5        -0.5            ]       [             2                   ]
[            0.5        -0.5      ]   x   [                   3             ]
[                  0.5        -0.5]       [                         4       ]
[                        0.5      ]       [                               5 ]
[                              0.5]       [                                 ]
```

To solve $u'(x) = f(x), \quad u(1) = 0$:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & & 3 & & 5 \\
 & & 4 & & 8 & \\
 & & & 6 & & 10 \\
 & & & & 8 & \\
 & & & & & 10
\end{bmatrix} u =
\begin{bmatrix}
 & & & & & \\
 & 1 & & & & \\
 & & 1 & & & \\
 & & & 1 & & \\
 & & & & 1 & \\
 & & & & & 1
\end{bmatrix} f
$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & & & & \\
 & & 2 & & & \\
 & & & 3 & & \\
 & & & & 4 & \\
 & & & & & 5
\end{bmatrix} u =
\begin{bmatrix}
 & & & & & \\
1 & & -0.5 & & & \\
 & 0.5 & & -0.5 & & \\
 & & 0.5 & & -0.5 & \\
 & & & 0.5 & & -0.5 \\
 & & & & 0.5 &
\end{bmatrix} f
$$

Ultraspherical spectral method

(A very quick recap)

To solve $u'(x) + u(x) = f(x), \quad \int_{-1}^{1} u(x)\,dx = 0$:

```
[ 2          -2/3          -2/15      ]       [                                      ]
[       2            3              5 ]       [      1                               ]
[             5            8          ] u  =  [             1                        ] f
[                    7             10 ]       [                    1                 ]
[                          8          ]       [                           1          ]
[                                  11 ]       [                                  1   ]

[ 2          -2/3          -2/15      ]       [                                      ]
[ 1     1    -0.5                     ]       [ 1          -0.5                      ]
[      0.5    2    -0.5               ] u  =  [      0.5          -0.5               ] f
[            0.5     3     -0.5       ]       [            0.5           -0.5        ]
[                   0.5     4     -0.5]       [                   0.5           -0.5]
[                          0.5      5 ]       [                          0.5         ]
```

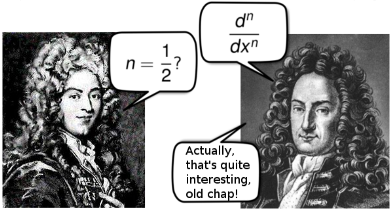
FRACTIONAL CALCULUS: BACKGROUND / HISTORY

Fractional calculus: Background / History
(300 hundred years in 30 seconds)

- History
  - L'Hôpital, Leibniz, Euler, Lacroix, Laplace, Fourier, Abel, Liouville, Riemann, ...[1]
  - "This [$d^{1/2}x/dx^{1/2}$] is an apparent paradox from which, one day, useful consequences will be drawn." – Leibniz, 1695
- Mathematical intuition

$$\frac{d^n}{dx^n}x^m = \frac{m!}{(m-n)!}x^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}, \quad (J^n f)(x) = \frac{1}{(n-1)!}\int_{-a}^{x}(x-t)^{n-1}f(t)dt$$

- Physical interpretation
  - Non-Gaussian random walks & heavy-tailed distributions. Memory processes.
- Application areas
  - Epidemiology
  - Finance
  - Physics
  - Porous media, ...



[1] See Bertram Ross, *The Development of Fractional Calculus 1695–1900*, (1997) for an excellent discussion of the history of fractional calculus.

FRACTIONAL INTEGRAL EQUATIONS

Fractional integral definition[2]:

$$ {}_aQ_x^\mu f(x) := \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}}\, dt. $$

Fractional integral of monomials:

$$ {}_0Q_x^\mu x^n = \frac{n!}{\Gamma(n+\mu+1)} x^{n+\mu}. $$

Fractional integral of exponentials and trigonometric functions:

$$ {}_{-\infty}Q_x^\mu e^{nx} = n^{-\mu} e^{nx} \quad \text{and} \quad {}_{-\infty}Q_x^\mu \sin(nx) = n^{-\mu}\sin(nx - \mu\pi/2). $$

Fractional integral of weighted Jacobi polynomials ("polyfractinomials"):

$$ {}_{-1}Q_x^\mu \left[(1+\diamond)^\beta P_n^{(\alpha,\beta)}\right](x) = \frac{B(\beta+n+1,\mu)}{\Gamma(\mu)}(1+x)^{\beta+\mu} P_n^{(\alpha-\mu,\beta+\mu)}(x). $$

[2] These are *left-sided* integrals. One can also define *right-sided* integrals.

Fractional integral of weighted Jacobi polynomials ("polyfractinomials"):

$$ {}_{-1}Q_x^{\mu}\left[(1+\diamond)^{\beta}P_n^{(\alpha,\beta)}\right](x) = \frac{B(\beta+n+1,\mu)}{\Gamma(\mu)}(1+x)^{\beta+\mu}P_n^{(\alpha-\mu,\beta+\mu)}(x). $$

The following conversions are also useful:

$$ \begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}P_n^{(\alpha,\beta+1)}(x)+\frac{n+\alpha}{2n+\alpha+\beta+1}P_{n-1}^{(\alpha,\beta+1)}(x), \\ &= \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}P_n^{(\alpha+1,\beta)}(x)-\frac{n+\beta}{2n+\alpha+\beta+1}P_{n-1}^{(\alpha+1,\beta)}(x), \\ (1+x)P_n^{(\alpha,\beta+1)}(x) &= \frac{2n+2}{2n+\alpha+\beta+2}P_{n+1}^{(\alpha,\beta)}(x)+\frac{2n+2\beta}{2n+\alpha+\beta+2}P_n^{(\alpha,\beta)}(x). \end{aligned} $$

- Combining these formulae leads to two important special case:

$$ {}_{-1}Q_x^{1/2} P_n(x) = \frac{\sqrt{1+x}}{\sqrt{\pi}(n+1/2)} \Big(U_n(x) - U_{n-1}(x) \Big) $$

$$ {}_{-1}Q_x^{1/2} \left[\sqrt{1+\diamond}\, U_n \right](x) = \frac{\sqrt{\pi}}{2} \Big(P_{n+1}(x) + P_n(x) \Big) $$

- Therefore half-integration is a banded operator between these spaces.

Abel integral equation:

$$\lambda u(x) + \int_{-1}^{x} \frac{u(t)}{\sqrt{x-t}}\,dt = e(x) + \sqrt{1+x}f(x)$$

Ansatz

$$u(x) = \sum_{n=0}^{\infty} a_n P_n(x) + \sqrt{1+x}\sum_{n=0}^{\infty} b_n U_n(x)$$

Leads to the (infinite dimensional) linear system

$$\begin{pmatrix} \lambda I & Q^{1/2}_{cheb_2} \\ Q^{1/2}_{leg} & \lambda I \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix}$$

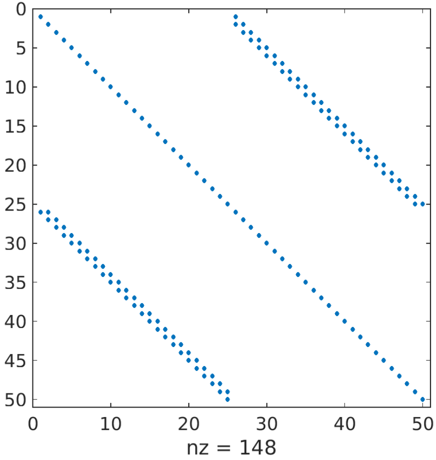
where

$$e(x) = \sum_{n=0}^{\infty} e_n P_n(x), \qquad f(x) = \sum_{n=0}^{\infty} f_n U_n(x)$$

$$\lambda u(x) + \int_{-1}^{x} \frac{u(t)}{\sqrt{x-t}} dt = e(x) + \sqrt{1+x} f(x)$$

$$A \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} := \begin{pmatrix} \lambda I & Q^{1/2}_{cheb_2} \\ Q^{1/2}_{leg} & \lambda I \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix}$$

- The operator/matrix A is block-banded.
- Interlacing the coefficients like $[a_0, b_0, a_1, b_1, \ldots]^T$ gives a tridiagonal matrix.
- Non-constant coefficients also work, but will increase bandwidth.



- If e and f are analytic, convergence is geometric.

$$\lambda u(x) + \int_{-1}^{x} \frac{u(t)}{\sqrt{x-t}} dt = e(x) + \sqrt{1+x} f(x)$$

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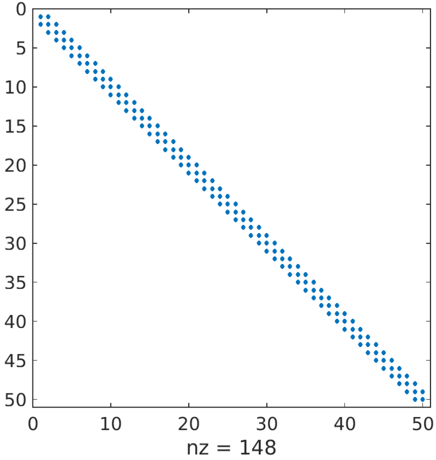

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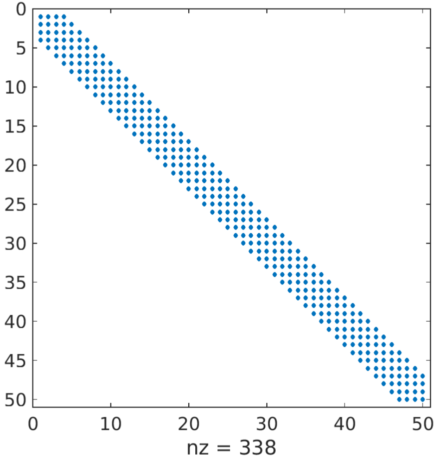

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- Interlacing the coefficients like $[a_0, b_0, a_1, b_1, \ldots]^T$ gives a tridiagonal matrix.
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$$\lambda u(x) + \int_{-1}^{x} \frac{u(t)}{\sqrt{x-t}} dt = e(x) + \sqrt{1+x} f(x)$$

$$A \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} := \begin{pmatrix} \lambda I & Q_{cheb_2}^{1/2} \\ Q_{leg}^{1/2} & \lambda I \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix}$$

- The operator/matrix A is block-banded.
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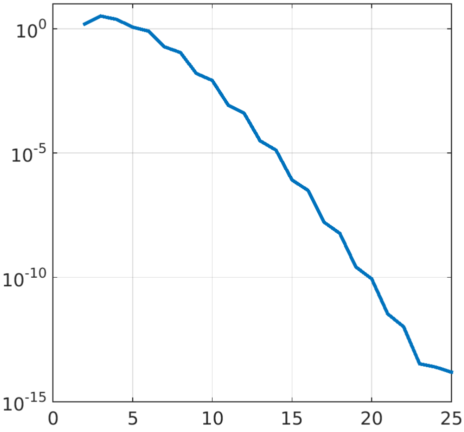


- If e and f are analytic, convergence is geometric.

FRACTIONAL DIFFERENTIAL EQUATIONS (FDEs)

There are then two main definitions of the half-derivative:

Riemann–Liouville:

$$ {}^{RL}_{\ a}D_x^{1/2} f(x) := D\ {}_aQ_x^{1/2} f(x) = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_a^x \frac{f(t)}{\sqrt{x-t}}\, dt. $$

Caputo:

$$ {}^{C}_{a}D_x^{1/2} f(x) := {}_aQ_x^{1/2} D f(x) = \frac{1}{\sqrt{\pi}} \int_a^x \frac{f'(t)}{\sqrt{x-t}}\, dt. $$

Chebyshev and Legendre polynomials satisfy:[3]

$$\frac{d}{dx}T_n(x) = nU_{n-1}(x) \quad \text{and} \quad \frac{d}{dx}P_n(x) = 2C_{n-1}^{(3/2)}(x)$$

Combining with previous results gives (derivation omitted):

$$ {}_{-1}^{RL}D_x^{1/2}P_n(x) = \frac{1}{\sqrt{1+x}\sqrt{\pi}}\Big(U_n(x) + U_{n-1}(x)\Big)$$

$$ {}_{-1}^{RL}D_x^{1/2}\left[\frac{T_n}{\sqrt{1+\diamond}}\right](x) = \frac{\sqrt{3}}{2}\Big(C_{n-1}^{(3/2)}(x) - C_{n-2}^{(3/2)}(x)\Big)$$

Here the output spaces aren't quite the same as the input, but it's OK because

$$2T_n(x) = U_n(x) - U_{n-2}(x) \quad \text{and} \quad (2n+1)P_n(x) = C_n^{(3/2)}(x) - C_{n-2}^{(3/2)}(x)!$$

[3] Here $C_n^{(\lambda)}(x)$ are Ultraspherical or "Gegenbauer" polynomials.

Fractional differential equation:

$$\lambda u(x) + {}_{-1}D_x^{1/2} u(x) = e(x) + \frac{1}{\sqrt{1+x}} f(x)$$

Ansatz

$$u(x) = \sum_{n=0}^{\infty} a_n P_n(x) + \frac{1}{\sqrt{1+x}} \sum_{n=0}^{\infty} b_n T_n(x)$$

Leads to the (infinite dimensional) linear system

$$\begin{pmatrix} \lambda S_1 & D_{cheb_1}^{1/2} \\ D_{leg}^{1/2} & \lambda S_2 \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix}$$

where

$$e(x) = \sum_{n=0}^{\infty} e_n C_n^{(3/2)}(x), \qquad f(x) = \sum_{n=0}^{\infty} f_n U_n(x)$$

BOUNDARY CONDITIONS?!

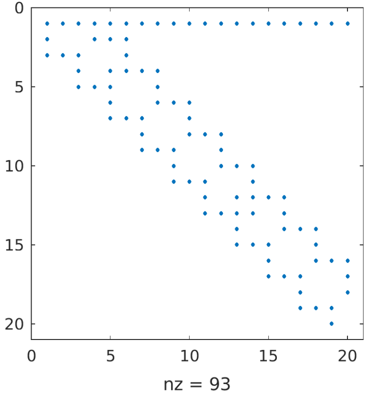
$$\lambda u(x) + {}_{-1}D_x^{1/2} u(x) = e(x) + \frac{1}{\sqrt{1+x}} f(x), \qquad x \in [-1, 1]$$

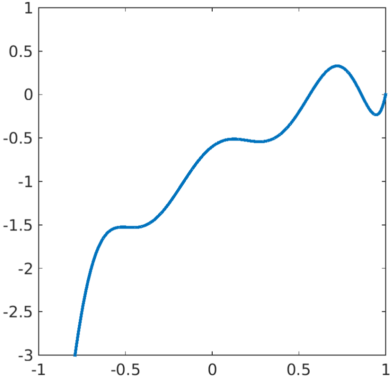
- We can't apply half a boundary condition - so do we apply 0 or 1?
- Notice that the half-derivative has the non-trivial kernel

$${}^{RL}_{-1}D_x^{1/2} \frac{1}{\sqrt{1+x}} = 0.$$

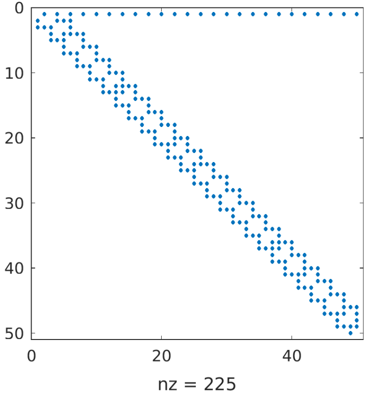
- Since the kernel is in our basis we need a boundary condition.
- But what conditions can we apply? Dirichlet at $x = -1 \rightarrow$ ill-posed
- To keep things simple, let's consider $u(1) = 0$ & $|u| < \infty$

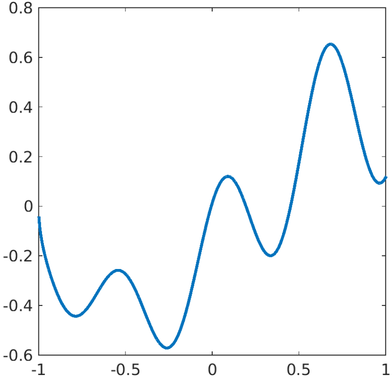
$$u(1) = \sum_{n=0}^{\infty} a_n P_n(1) + \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} b_n T_n(1) = 0$$





$$|u(-1)| < \infty \to \sum_{n=0}^{\infty} b_n T_n(1) = 0$$





- Things I didn't talk about:
  - Higher-order derivatives
  - Computing Legendre coefficients
  - Solving the infinite-dimensional banded linear systems
  - Caputo definition of fractional derivatives
  - Existing methods?
- Extensions:
  - Non-constant coefficients
  - Non-linear problems
  - Fractional partial differential equations
  - Non-half integer order equations?
  - Two-sided derivatives?

Moral of the story:

banded differential operator

$+$

banded conversion matrices

$=$

fast algorithm

THE END - THANKS!