A unified theory for turbulent wake flows described by eddy viscosity

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March 23, 2016

Outline

Review of previous work

- Classical laminar wake-Goldstein.
- Self-propelled laminar wake-Birkhoff and Zorantello.
- Two-fluid laminar wake-Herzynski, Weidman and Burde.
- Turbulent planar wake-Tennekes and Lumley.

Similarity solutions are obtained when the eddy viscosity is a power law of the distance along the wake and when the kinematic viscosity is neglected.

Mathematical model

- Flow = mean motion + eddying motion.
- Implement averaging techniques.
- Time averages of fluctuations are zero but the time averages of products of fluctuations are non-zero.
- Non-zero terms are expressed in terms of Reynolds stresses.
- Eddy viscosity closure model is implemented.
- Boundary layer theory is applied.
- Dimensionless equations are expressed in terms of the velocity deficit: $\overline{v}_x = 1 - \overline{w}(x, y)$.

Mathematical model

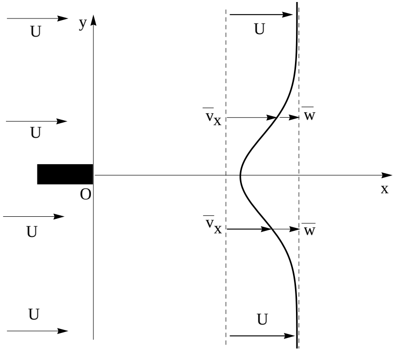

Figure: Two-dimensional wake behind a slender symmetric body aligned with a uniform mainstream flow. The origin of the coordinate system is at the trailing edge of the object.

Mathematical model

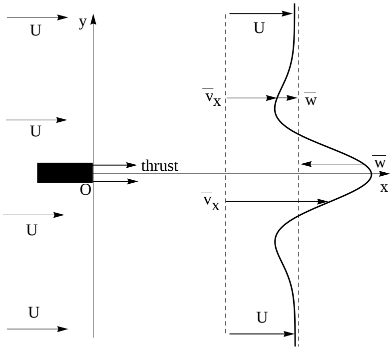

Figure: Two-dimensional wake behind a slender symmetric self-propelled body. The mean velocity deficit is negative in a neighbourhood of the x-axis.

Mathematical model

We assume that we are sufficiently far downstream such that powers and products can be neglected. The governing equations are

$$-\frac{\partial \overline{w}}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} = 0, \tag{1}$$

$$\frac{\partial \overline{w}}{\partial x} = \frac{\partial}{\partial y}\left(E\left(x, y\right) \frac{\partial \overline{w}}{\partial y} \right), \tag{2}$$

where the dimensionless effective viscosity E is

$$E\left(x, y\right) = \frac{\nu}{\nu + \nu_{T_0}} + \frac{\nu_T}{\nu + \nu_{T_0}}. \tag{3}$$

The boundary conditions for $x \geq 0$ are

$$\overline{w}(x, \pm\infty) = 0, \quad \frac{\partial \overline{w}}{\partial y}(x, \pm\infty) = 0, \tag{4}$$

$$\frac{\partial \overline{w}}{\partial y}(x, 0) = 0, \quad \overline{v}_y(x, 0) = 0. \tag{5}$$

Mathematical model

In terms of a stream function we have

$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial}{\partial y}\left(E\left(x,y\right)\frac{\partial^2\psi}{\partial y^2}\right). \tag{6}$$

The boundary conditions, for $x \geq 0$, are

$$\frac{\partial\psi}{\partial y}(x,\pm\infty) = 0, \quad \frac{\partial^2\psi}{\partial y^2}(x,\pm\infty) = 0, \tag{7}$$

$$\frac{\partial\psi}{\partial x}(x,0) = 0, \quad \frac{\partial^2\psi}{\partial y^2}(x,0) = 0. \tag{8}$$

The conserved quantities are

$$\int_{-\infty}^{\infty}\frac{\partial\psi}{\partial y}dy = D, \tag{9}$$

for the classical wake and for the wake of a self-propelled body

$$\int_{-\infty}^{\infty}y^2\frac{\partial\psi}{\partial y}dy = K. \tag{10}$$

Conserved vectors

- Conservation laws are of the form

$$D_1 T^1 + D_2 T^2 \Big|_{\text{PDE}} = 0. \tag{11}$$

- We used the multiplier method in order to derive the conserved vectors.

$$\Lambda^j F_j(x, \psi, \psi_{(1)}, \dots, \psi_{(k)}) = D_j T^j. \tag{12}$$

- In terms of the stream function we obtained

$$T^1 = y^2 \psi_y, \quad T^2 = \left(-y^2 \psi_{yy} + 2y\psi_y - 2\psi\right) E(x). \tag{13}$$

$$T^1 = y\psi_y, \quad T^2 = -yE(x)\psi_{yy} + E(x)\psi_y, \tag{14}$$

$$T^1 = \psi_y, \quad T^2 = -E\,(x, y)\,\psi_{yy}. \tag{15}$$

Conserved vectors

- And for the velocity components we had

$$T^1 = \left(y^2 - 2\int_0^x E(\alpha)d\alpha \right) \overline{w},$$

$$T^2 = -y^2 E(x)\overline{w}_y + 2yE(x)\overline{w} + 2\int_0^x E(\alpha)d\alpha\overline{v}. \tag{16}$$

$$T^1 = y\overline{w}, \quad T^2 = -yE(x)\overline{w}_y + E(x)\overline{w}. \tag{17}$$

$$T^1 = \overline{w}, \quad T^2 = -E\left(x,y\right)\overline{w}_y. \tag{18}$$

$$T^1 = -\overline{w}, \quad T^2 = \overline{v}. \tag{19}$$

Classical wake

- The elementary conserved vector

$$T^1 = \psi_y, \quad T^2 = -E\,(x,y)\,\psi_{yy}, \tag{20}$$

generates the conserved quantity.

- The Lie point symmetry is

$$X = \xi^1(x)\frac{\partial}{\partial x} + \xi^2(x,y)\frac{\partial}{\partial y} + \eta(x)\frac{\partial}{\partial \psi}, \tag{21}$$

provided $E(x,y)$ satisfies

$$E(x,y)\frac{\partial^2 \xi^2}{\partial y^2} - \frac{\partial \xi^2}{\partial x} = 0, \tag{22}$$

$$\xi^1(x)\frac{\partial E}{\partial x} + \xi^2(x,y)\frac{\partial E}{\partial y} = \left(2\frac{\partial \xi^2}{\partial y} - \frac{d\xi^1}{dx}\right)E(x,y). \tag{23}$$

Classical wake

- We consider $E = E(x)$ only.
- The Lie point symmetry becomes

$$X = \frac{2\int_0^x E(x')dx'}{E(x)}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{24}$$

- The invariant solution is

$$\psi(x,y) = F(\xi), \tag{25}$$

where

$$\xi = \frac{y}{(2\int_0^x E(x')dx')^{1/2}}, \tag{26}$$

and $F(\xi)$ satisfies

$$\frac{d^3F}{d\xi^3} + \xi\frac{d^2F}{d\xi^2} + \frac{dF}{d\xi} = 0, \tag{27}$$

Classical wake

- subject to

$$\frac{dF}{d\xi}(\pm\infty) = 0, \quad \frac{d^2F}{d\xi^2}(\pm\infty) = 0, \tag{28}$$

$$\frac{d^2F}{d\xi^2}(0) = 0, \qquad F(0) = 0. \tag{29}$$

- The solution is

$$\psi(x, y) = \frac{D}{\sqrt{2\pi}} \int_0^{\xi} \exp\left[-\frac{\xi^{*2}}{2}\right] d\xi^*. \tag{30}$$

Wake of a self-propelled body

- The conserved vector is

$$T^1 = y^2\psi_y, \quad T^2 = \left(-y^2\psi_{yy} + 2y\psi_y - 2\psi\right) E(x). \tag{31}$$

- Lie point symmetry associated with conserved vector

$$X = \frac{1}{E(x)}\left[2\int_0^x E(\alpha)d\alpha\right]\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\psi\frac{\partial}{\partial \psi}. \tag{32}$$

- Invariant solution

$$\psi(x,y) = \frac{F(\xi)}{2\int_0^x E(\alpha)d\alpha}, \tag{33}$$

where

$$\xi(x,y) = \frac{y}{\left(2\int_0^x E(\alpha)d\alpha\right)^{1/2}}, \tag{34}$$

and $F(\xi)$ satisfies

$$\frac{d^3F}{d\xi^3} + \xi\frac{d^2F}{d\xi^2} + 3\frac{dF}{d\xi} = 0, \tag{35}$$

Wake of a self-propelled body

- subject to

$$\frac{dF}{d\xi}(\pm\infty) = 0, \quad \frac{d^2F}{d\xi^2}(\pm\infty) = 0, \tag{36}$$

$$F(0) = 0, \quad \frac{d^2F}{d\xi^2}(0) = 0. \tag{37}$$

- The solution is

$$\psi(x,y) = -\frac{K}{2\sqrt{2\pi}} \frac{1}{\left[2\int_0^x E(\alpha)d\alpha\right]} \xi \exp\left[-\frac{\xi^2}{2}\right]. \tag{38}$$

Discovery of the 'odd' wake

- The conserved vector

$$T^1 = y\psi_y, \quad T^2 = -yE(x)\psi_{yy} + E(x)\psi_y, \tag{39}$$

led to the discovery of the odd wake! Which we called the combination wake...

- Lie point symmetry:

$$X = \frac{1}{E(x)}\left[2\int_0^x E(\alpha)d\alpha\right]\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \psi\frac{\partial}{\partial \psi}. \tag{40}$$

- Invariant solution:

$$\psi(x,y) = \frac{F(\xi)}{\left(2\int_0^x E(\alpha)d\alpha\right)^{1/2}}, \tag{41}$$

where

$$\xi(x,y) = \frac{y}{\left(2\int_0^x E(\alpha)d\alpha\right)^{1/2}}, \tag{42}$$

Discovery of the 'odd' wake

and $F(\xi)$ satisfies

$$\frac{d^3F}{d\xi^3} + \xi\frac{d^2F}{d\xi^2} + 2\frac{dF}{d\xi} = 0, \tag{43}$$

subject to

$$\frac{dF}{d\xi}(\pm\infty) = 0, \quad \frac{d^2F}{d\xi^2}(\pm\infty) = 0, \tag{44}$$

$$F(0) = 0. \tag{45}$$

Discovery of the 'odd' wake

- The solution is

$$\psi(x,y) = \frac{S}{2\sqrt{\pi}\left(\int_0^x E(\alpha)d\alpha\right)^{1/2}}\left[1 - \exp\left(-\frac{\xi^2}{2}\right)\right]. \quad (46)$$

- Used similar physical arguments to obtain the constants in the Lie point symmetry.
- The same boundary conditions are satisfied at $\pm\infty$.
- However, the velocity deficit is not a max on the axis of the wake- in fact it is zero.

The combination wake

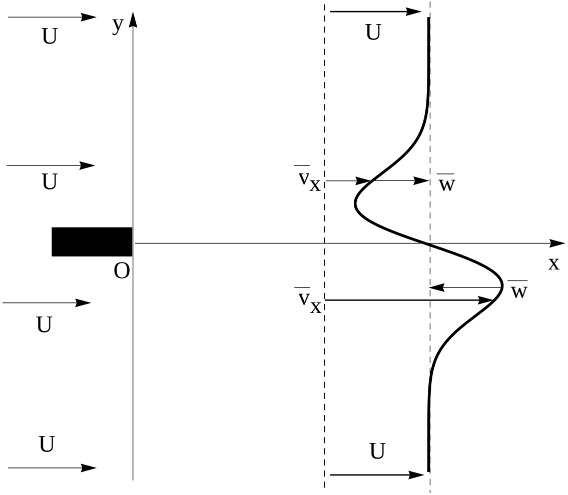

Figure: Two-dimensional combination wake behind a slender symmetric body.

Summary

- Classical wake

$$\psi(x,y) = \frac{D}{\sqrt{2\pi}} \int_0^{\xi} \exp\left[-\frac{\xi^{*2}}{2}\right] d\xi^* \tag{47}$$

- Combination wake

$$\psi(x,y) = \frac{S}{\sqrt{2\pi}} \frac{1}{\left[2\int_0^x E(\alpha)d\alpha\right]^{1/2}} \left(1 - \exp\left[-\frac{\xi^2}{2}\right]\right) \tag{48}$$

- Wake of a self-propelled body

$$\psi(x,y) = -\frac{K}{2\sqrt{2\pi}} \frac{1}{\left[2\int_0^x E(\alpha)d\alpha\right]} \xi \exp\left[-\frac{\xi^2}{2}\right] \tag{49}$$

Mathematical link

- Surprisingly all these solutions are linked!
- For $E = E(x)$ the governing equation is linear:

$$\frac{\partial^2 \psi}{\partial x \partial y} = E(x) \frac{\partial^3 \psi}{\partial y^3}. \tag{50}$$

- Therefore, ψ_n where

$$\psi_n(x,y) = \frac{\partial^n \psi}{\partial y^n}, \quad n \geq 1, \tag{51}$$

are also solutions! But they don't necessarily satisfy the BCs. We have

$$\frac{\partial^2 \psi_n}{\partial y^2}(x, \pm\infty) = 0, \quad \frac{\partial \psi_n}{\partial y}(x, \pm\infty) = 0, \tag{52}$$

$$\frac{\partial \psi_n}{\partial x}(x,0) = 0, \quad \frac{\partial^2 \psi_n}{\partial y^2}(x,0) = 0, \quad \text{n even}. \tag{53}$$

Mathematical link

- We let

$$\psi(x,y) = \alpha_1 \psi_n(x,y) + \alpha_2(x). \tag{54}$$

- For $n = 1$ we can recover the solution for the combination wake.
- For $n = 2$ we can recover the solution for the wake of a self-propelled body.
- The combination wake provided the link between all the solutions!

Conclusions

- Conservation laws lead to the discovery of the combination wake.
- All wake problems are mathematically linked which highlights the importance of using modelling with the symmetries.