On the zeros of Meixner and Meixner-Pollaczek polynomials

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Orthogonal polynomials

To define families of orthogonal polynomials, we use a scalar product

$$\langle f, g\rangle := \int_a^b f(x)g(x)\ d\phi(x),$$

positive measure $d\phi(x)$ supported on $[a,b]$, $a, b \in \mathbb{R}$.

A sequence of real polynomials $\{p_n\}_{n=0}^N$, $N \in \mathbb{N} \cup \{\infty\}$, is orthogonal on (a,b) with respect to $d\phi(x)$ if

$$\langle p_n, p_m\rangle = 0 \text{ for } m = 0, 1, \ldots, n-1.$$

Orthogonal polynomials

- If $d\phi(x)$ is absolutely continuous and $d\phi(x) = w(x)dx$,

$$\int_a^b p_n(x)p_m(x)w(x)dx = 0 \text{ for } m = 0, 1, \ldots, n-1$$

$\{p_n\}$ is orthogonal on (a, b) w.r.t. the weight $w(x) > 0$.

- If the weight is discrete and $w_j = w(j)$, $j \in \mathbb{L} \subset \mathbb{Z}$,

$$\sum_{j\in\mathbb{L}} p_n(j)p_m(j)\; w_j \;=\; 0 \text{ for } m = 0, 1, \ldots, n-1$$

and the sequence $\{p_n\}$ is **discrete orthogonal**.

In the classical case: $\mathbb{L} = \{0, 1, \ldots, N\}$.

Properties of orthogonal polynomials

(i) **Three-term recurrence relation**

$$(x - B_n)p_{n-1}(x) = A_n p_n(x) + C_n p_{n-2}(x), \ \ n \geq 1$$

$$p_{-1}(x) = 0; \quad A_n, B_n, C_n \in \mathbb{R}; \quad A_{n-1}C_n > 0, \ \ n = 1, 2, \ldots;$$

(ii) p_n has n real, distinct zeros in (a, b);

(iii) **Classic interlacing of zeros**
The zeros of p_n and p_{n-1} separate each other:
$a < x_{n,1} < x_{n-1,1} < x_{n,2} < \cdots < x_{n-1,n-1} < x_{n,n} < b$.

Orthogonality and quasi-orthogonality

- Polynomials are orthogonal for specific values of their parameters, e.g.
  Jacobi polynomials ($P_n^{\alpha,\beta}$):
  orthogonal on $[-1,1]$ w.r.t $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ for $\alpha, \beta > -1$.
- Deviation from restricted values of the parameters results in zeros departing from interval of orthogonality
- Question: Do polynomials with "shifted" parameters retain some form of orthogonality that explains the amount of zeros that remain in the interval of orthogonality?

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Quasi-orthogonality (Riesz, 1923)

A sequence of polynomials $\{R_n\}_{n=0}^{N}$ is **quasi-orthogonal** of order k with respect to $w(x)$ on $[a, b]$ if

$$\int_a^b x^m R_n(x) w(x) dx \begin{cases} = 0 & \text{for} \quad m = 0, 1, \ldots, n-k-1 \\ \neq 0 & \text{for} \quad m = n-k. \end{cases}$$

Note that $n = k+1, k+2, \ldots$.

Preliminary results (Shohat, Brezinski et al)

Lemma 1

Let $\{p_n\}$ be orthogonal on $[a, b]$ with respect to $w(x)$. A necessary and sufficient condition for a polynomial R_n to be quasi-orthogonal of order k on $[a, b]$ with respect to $w(x)$, is that

$$R_n(x) = c_0 p_n(x) + c_1 p_{n-1}(x) + \cdots + c_k p_{n-k}(x)$$

where the c_i's are numbers which can depend on n and $c_0 c_k \neq 0$.

Lemma 2

If $\{R_n\}$ are real polynomials that are quasi-orthogonal of order k with respect to $w(x)$ on an interval $[a, b]$, then at least $(n-k)$ zeros of $R_n(x)$ lie in the interval $[a, b]$.

Meixner polynomials (Josef Meixner, 1934)

$$M_n(x;\beta,c) = (\beta)_n \sum_{k=0}^{n} \frac{(-n)_k(-x)_k(1-\frac{1}{c})^k}{(\beta)_k k!}$$

$\beta, c \in \mathbb{R},\ \beta \neq -1, -2, \ldots, -n+1,\ c \neq 0.$

$(\ )_k$ is the Pochhammer symbol

$$\begin{aligned} (a)_k &= a(a+1)...(a+k-1),\ k \geq 1 \\ (a)_0 &= 1 \text{ when } a \neq 0 \end{aligned}$$

Meixner polynomials

For $0 < c < 1$, $\beta > 0$,

$$\sum_{j=0}^{\infty} \frac{c^j (\beta)_j}{j!} M_m(j;\beta,c) M_n(j;\beta,c) = 0, m = 1, 2, \dots, n-1,$$

hence the zeros are real, distinct and in $(0, \infty)$.

- $\frac{c^j (\beta)_j}{j!}$ constant on $(j, j+1), j = 0, 1, 2, \dots;$
- zeros are separated by mass points $j = 0, 1, 2, \dots$.

Difference equation

Meixner polynomials satisfy the difference equation:

$$c(x+\beta)M_n(x+1;\beta,c) = \Big(n(c-1)+x+(x+\beta)c\Big)M_n(x;\beta,c) - xM_n(x-1;\beta,c).$$

Krasikov, Zarkh (2009): Suppose $p_n(x)$ satisfies

$$p_n(x+1) = 2A(x)p_n(x) - B(x)p_n(x-1)$$

and $B(x) > 0$ for $x \in (a,b)$, then $M(p_n) > 1$.
$M(p_n) \equiv$ minimum distance between the zeros of $p_n(x)$.

- True for Hahn, Meixner, Krawtchouk and Charlier polynomials;
- Hahn polynomials: Levit (1967);
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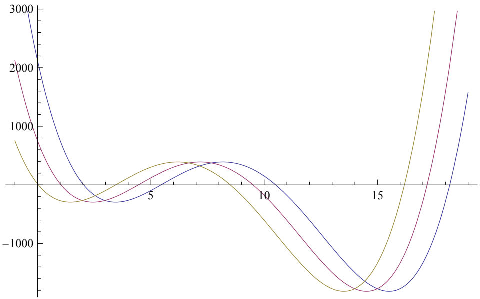
As a consequence:
Zeros of $p_n(x-1), p_n(x)$ and $p_n(x+1)$ interlace.



Zeros of $M_4(x-1,5;0.45), M_4(x,5;0.45)$ and $M_4(x+1,5;0.45)$.

Jordaan, Tookos, AJ (2011)

Let $0 < \beta < 1, 0 < c < 1$. By iterating the recurrence relation

$$M_n(x;\beta-1,c) = M_n(x;\beta,c) - nM_{n-1}(x;\beta,c),$$

we obtain

$$M_n(x;\beta-k,c) = c_0 M_n(x;\beta,c) + c_1 M_{n-1}(x;\beta,c) + \cdots + c_k M_{n-k}(x;\beta,c)$$

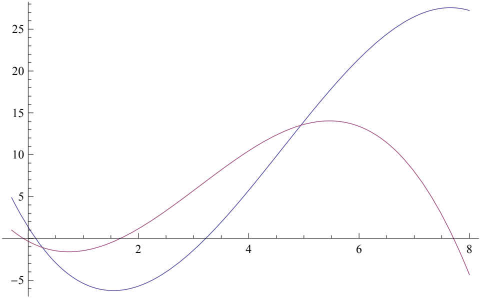
and

- $M_n(x;\beta-k,c)$ is quasi-orthogonal of order k for $k \in \{1,2,\ldots n-1\}$;
- at least $n-k$ zeros remain in $(0,\infty)$.

To obtain relations necessary to prove our results, we use a Maple program by Vidunas.

Quasi-orthogonality of order 1

Theorem: If $0 < c < 1$ and $0 < \beta < 1$, then the smallest zero of $M_n(x; \beta - 1, c)$ is negative.



Zeros of $M_3(x, 0.4; 0.6)$ and $M_3(x, 0.4 - 1; 0.6)$.

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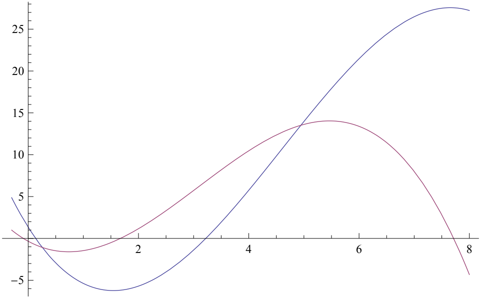

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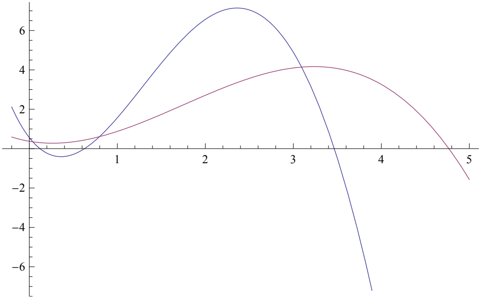
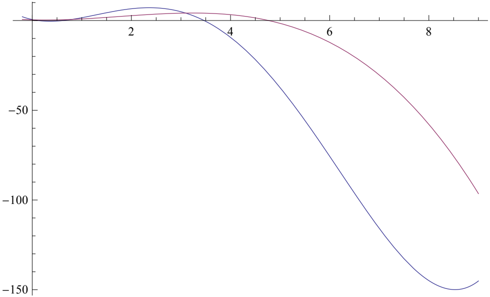
Quasi-orthogonality of order 2

Theorem: If $0 < c < 1$, $0 < \beta < 1$ and $n > \frac{\beta-2}{c-1}$ then all the zeros of $M_n(x; \beta-2, c)$ are nonnegative and simple.

For $\beta = 0.5$ and $c = 0.6$, $\frac{\beta-2}{c-1} = 3.75$.



Zeros of $M_4(x, 0.5-2; 0.6)$ and $M_3(x, 0.5-2; 0.6)$.



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Meixner-Pollaczek polynomials

Definition of the (monic) Meixner polynomials,

$$M_n(x;\beta,c) = (\beta)_n \left(\frac{c}{c-1}\right)^n \sum_{k=0}^{n} \frac{(-n)_k(-x)_k(1-\frac{1}{c})^k}{(\beta)_k k!}$$

Let $c = e^{2i\phi}, x = -\lambda - ix$ and $\beta = 2\lambda$, to obtain the Meixner-Pollaczek polynomials

$$P_n^{\lambda}(x;\phi) = i^n(2\lambda)_n \left(\frac{e^{2i\phi}}{e^{2i\phi}-1}\right)^n \sum_{k=0}^{n} \frac{(-n)_k(\lambda+ix)_k(1-\frac{1}{e^{2i\phi}})^k}{(2\lambda)_k k!}.$$

For $n \in \mathbb{N}$, $\lambda > 0$, $0 < \phi < \pi$, $P_n^{\lambda}(x;\phi)$ are orthogonal on $(-\infty,\infty)$ w.r.t. $e^{(2\phi-\pi)x}|\Gamma(\lambda+ix)|^2$.

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Johnston, Jordaan, AJ (2016)

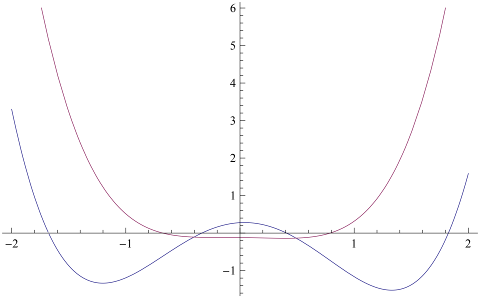
Theorem
For $0 < \lambda < 1$ and $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ fixed,
the polynomial $P_n^{\lambda-k}(x;\phi)$ is quasi-orthogonal of order $2k$ with respect to $e^{(2\phi-\pi)x}|\Gamma(\lambda+ix)|^2$ on $(-\infty,\infty)$ and therefore has at least $n-2k$ real zeros.

Contiguous relations are used to find

$$P_n^{\lambda-1}(x;\phi) = P_n^{\lambda}(x;\phi) - n\cot\phi P_{n-1}^{\lambda}(x;\phi) + \frac{n(n-1)}{4\sin^2\phi}P_{n-2}^{\lambda}(x;\phi)$$

By iteration, $P_n^{\lambda-k}(x;\phi)$ can be written as a linear combination of $P_n^{\lambda}(x,\phi), P_{n-1}^{\lambda}(x,\phi), \ldots, P_{n-2k}^{\lambda}(x,\phi)$ and we can apply Lemmas 1 and 2.

Johnston, Jordaan, AJ (2016)



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Thank you