Characterizing properties of generalized Freud polynomials

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Outline

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§18.32 OP's with Respect to Freud Weights

A *Freud weight* is a weight function of the form

$$\textbf{18.32.1} \qquad w(x) = exp(-Q(x)), \quad -\infty < x < \infty$$

where $Q(x)$ is real, even, non-negative, and continuously differentiable. Of special interest are the cases $Q(x) = x^{2m}$, $m = 1, 2, \ldots$. **No explicit expressions for the corresponding OP's are available**. However, for asymptotic approximations in terms of elementary functions for the OP's, and also for their largest zeros, see Levin and Lubinsky [2001] and Nevai [1986]. For a uniform asymptotic expansion in terms of Airy functions for the OP's in the case x^4 see Bo and Wong [1999].

Orthogonal polynomial sequences

- Given $\{\mu_n\} \in \mathbb{R}$, we define the moment functional $\mathfrak{L} : x^n \to \mu_n$ on the linear space of polynomials $\mathcal{P}$. Assume $\mu_0 = \mathfrak{L}(1) = 1$. The inner product $\langle \cdot, \cdot \rangle$ for the functional $\mathcal{L}$ is given by

$$\langle P_m(x), P_n(x) \rangle = \mathcal{L}\left(P_m(x) P_n(x)\right)$$

- Monic polynomials $\{P_n(x)\}_{n=0}^{\infty}$ orthogonal w.r.t. a moment functional $\mathfrak{L}$ related to an absolutely continuous Borel measure μ on $\mathbb{R}$; $d\mu(x) = w(x)\, dx; w(x) > 0$:

$$\mathfrak{L}\left(P_m(x)\, P_n(x)\right) = \langle P_m, P_n \rangle = \int_{\mathbb{R}} P_m(x)\, P_n(x)\ \ d\mu(x) = h_n\, \delta_{mn},$$

where the normalization constant $h_n > 0$ and δ_{mn} is the Kronecker delta.

- Monic orthogonal polynomials $P_n(x)$ satisfy

$$P_{-1}(x) = 0,\; P_0(x) = 1,$$
$$P_{n+1}(x) = (x - \alpha_n)\, P_n(x) - \beta_n\, P_{n-1}(x),$$
$$\alpha_n = \frac{\langle xP_n, P_n\rangle}{\langle P_n, P_n\rangle} \in \mathbb{R};\; \beta_n = \frac{\langle P_n, P_n\rangle}{\langle P_{n-1}, P_{n-1}\rangle} > 0, \beta_0 = 1,\; n \in \mathbb{N}_0,$$

and the constant:

$$h_n = \langle P_n, P_n\rangle = \|P_n\|^2 = \prod_{j=1}^{n} \beta_j.$$

- To construct $P_n(x)$ for $\mathfrak{L}$:

$$P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix},\; \Delta_n := \det(\mu_{i+j})_{i,j=0}^{n} > 0.$$

Classical orthogonal polynomials

- Classical weights satisfy Pearson's equation
$$\frac{d}{dx}(\sigma w) = \tau w, \tag{2.1}$$
with $\deg(\sigma) \leq 2$ and $\deg(\tau) = 1$, and bcs : $\sigma(x)\ w(x) = 0$ for $x = a$ and $x = b$.

p_n	$w(x)$	$\sigma(x)$	$\tau(x)$	interval
Hermite	$exp(-x^2)$	1	$-2x$	$(-\infty, \infty)$
Laguerre	$x^\alpha exp(-x),\ \alpha > -1$	x	$1+\alpha-x$	$(0, \infty)$
Jacobi	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$	$\beta-\alpha-(2+\alpha+\beta)x$	$[-1, 1]$

- p_n's are solutions of $Lp_n = \lambda_n p_n$ where L is a second order differential operator (Sturm-Liouville) [Bochner, 1929]
- Structural relation:
$$\sigma(x)\,(p_n(x))' = \sum_{j=n-1}^{n-r+1} A_{n,j} p_j(x), \quad r = \deg(\sigma) \tag{2.2}$$
- (2.2) *together with* $xp_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}$, *yields a first order recurrence equation for the recurrence coefficients* a_n *and* b_n, *which can be solved explicitly.*

Semi-classical orthogonal polynomials

- Semi-classical weights satisfy Pearson's equation (2.1) with $\deg(\sigma) > 2$ or $\deg(\tau) > 1$. [Hendriksen, van Rossum, 1977]

weight	$w(x)$	parameters	$\sigma(x)$	$\tau(x)$
-	$exp(-x^4)$	-	1	$-4x^3$
Airy	$exp(-\frac{1}{3}x^3 + tx)$	$t > 0$	1	$t - x^2$
Semi-classical Laguerre	$x^\lambda exp(-x^2 + tx)$	$\lambda > -1$	x	$1 + \lambda + tx - 2x^2$
Freud	$exp(-\frac{1}{4}x^4 - tx^2)$	$x, t \in \mathbb{R}$	1	$-2tx - x^3$
Generalized Freud	$\|x\|^{2\lambda+1}exp(-x^4 + tx^2)$	$\lambda > 0,\ x, t \in \mathbb{R}$	x	$2\lambda + 2 - 2tx^2 - x^4$

- p_n does not satisfy Sturm-Liouville differential equation.
- Structural relation

$$\sigma(x)p_n^{'}(x) = \sum_{j=n-s}^{n-r+1} A_{n,j}p_j(x), \quad \begin{cases} r = \deg(\sigma), \\ s = \max\{\deg(\sigma) - 1, \deg(\tau)\} \end{cases} . \tag{2.3}$$

(2.3) *and* $xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x),\ n \geq 0$, *yield second or higher order (non-linear) equations for the recurrence coefficients* a_n *and* b_n.

Example: $w(x) = \exp(-x^4)$ on $\mathbb{R}$ [Nevai, 1983]: $b_n = 0$ (symmetry);

$$4a_n^2\left(a_n^2 + a_n^2 + a_n^2\right) = n,\ \ n \geq 2, \quad a_0 = 1, \quad a_1^2 = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}, \text{ where } \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

The link to Painlevé equations

- **Some history**: The first non-linear recurrence equation – **Shohat**(1930's) and **Laguerre**, **Freud** (late 70's) and very recently recognized as discrete Painlevé equations by **Fokas, Its,** and **Kitaev**. Work by **Magnus** (relation between discrete and continuous Painlevé equations), **Witte**, **Clarkson**, **Van Assche**, **Nijhoff**, **Spicer**, **Chen** and **Ismail** extended theory with some more examples.
- **Some Discrete Painlevé eqns:**

$$(\text{d-P}_{\mathrm{I}}) \quad x_{n+1} + x_n + x_{n-1} = \frac{z_n + \gamma(-1)^n}{x_n} + \sigma$$

$$(\text{d-P}_{\mathrm{II}}) \quad x_{n+1} + x_{n-1} = \frac{x_n z_n + \gamma}{1 - x_n^2}$$

$$(\text{d-P}_{\mathrm{IV}}) \quad (x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - \kappa^2)(x_n^2 - \mu^2)}{(x_n + z_n)^2 - \gamma^2}$$

- The continuous fourth Painlevé equation (P_{IV})

$$\frac{d^2q}{dz^2} = \frac{1}{2q}\left(\frac{dq}{dz}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q}, \tag{3.1}$$

where A and B are constants, which are expressed in terms of parabolic cylinder (Hermite-Weber) functions.

Semi-classical Laguerre

Theorem (LB-WVA, 2012)

The coefficients $\alpha_n(t)$ and $\beta_n(t)$ in the three-term recurrence

$$L_{n+1}^{(\nu)}(x;t) = (x-\alpha_n)L_n^{(\nu)}(x;t) - \beta_n L_{n-1}^{(\nu)}(x;t);$$

associated with the the semi-classical Laguerre $w_\nu(x) = x^\nu \exp(-x^2+tx)$, $\nu > -1,\ x \in \mathbb{R}^+$ *are:*

$$(2\alpha_n - t)(2\alpha_{n-1} - t) = \frac{(2\beta_n - n)(2\beta_n - n - \nu)}{\beta_n},$$

$$2\beta_n + 2\beta_{n+1} - \alpha_n(2\alpha_n - t) = 2n+1+\nu.$$

For explicit formulations of α_n and β_n, see [CJ, 2014].

Discrete Painlevé and more semi-classical weights

Question: What semi-classical weights are related to discrete Painlevé equations? Which discrete Painlevé equations do we obtain?

- $w(x) = |x|^{\varrho} \exp(-x^4)$, $\varrho > -1$ on $\mathbb{R}$ is related to (d-$\mathrm{P_I}$). [Magnus, 1986].
- $w(x) = x^{\alpha} \exp(-x^2)$, $\alpha > -1$ on $\mathbb{R}^+$ is related to (d-$\mathrm{P_{IV}}$) [Sonin-type].
- $w(x;t) = x^{\alpha} \exp(-x^2 + tx)$, $\alpha > -1$ on $\mathbb{R}^+$ is related to $(\mathrm{P_{IV}})$ [GF-WVA-LZ, 2011].
- $W_{\lambda}(x) = |x|^{2\lambda+1} \exp(-x^4 + tx^2)$, $\lambda > -1$, $t, x \in \mathbb{R}$ related to (d-$\mathrm{P_I}$) and continuous $(\mathrm{P_{IV}})$ [LB-WVA, 2011, GF-WVA-LZ, 2012].

The recurrence coefficient related to Painlevé IV

Theorem. (LB-WVA, 2011; GF-WVA-LZ, 2012)

The recurrence coefficients $\beta_n(t;\lambda)$ in the three term recurrence

$$xS_n(x;t) = S_{n+1}(x;t) + \beta_n(t;\lambda)S_{n-1}(x;t)$$

associated with the weight W_λ satisfy the equation

$$\frac{d^2\beta_n}{dt^2} = \frac{1}{2\beta_n}\left(\frac{d\beta_n}{dt}\right)^2 + \tfrac{3}{2}\beta_n^3 - t\beta_n^2 + (\tfrac{1}{8}t^2 - \tfrac{1}{2}A_n)\beta_n + \frac{B_n}{16\beta_n}, \qquad (3.2)$$

where the parameters A_n and B_n are given by

$$\begin{array}{ll} A_{2n} = -2\lambda - n - 1, & A_{2n+1} = \lambda - n, \\ B_{2n} = -2n^2, & B_{2n+1} = -2(\lambda + n + 1)^2. \end{array}$$

Further $\beta_n(t)$ satisfies the non-linear difference equation

$$\beta_{n+1} + \beta_n + \beta_{n-1} = \tfrac{1}{2}t + \frac{2n + (2\lambda+1)[1-(-1)^n]}{8\beta_n}, \quad \text{–discrete } \mathrm{P_I}\ (\mathrm{dP_I}).$$

Remark: *(3.2)* $\equiv \mathrm{P_{IV}}$ *via the transformation* $\beta_n(t;\lambda) = \frac{1}{2}w(z)$, *with* $z = -\frac{1}{2}t$. *Hence*

$$\beta_{2n}(t;\lambda) = \tfrac{1}{2}w\big(z; -2\lambda - n - 1, -2n^2\big); \quad \beta_{2n+1}(t;\lambda) = \tfrac{1}{2}w\big(z; \lambda - n, -2(\lambda+n+1)^2\big),$$

with $z = -\frac{1}{2}t$, *where* $w(z;A,B)$ *satisfies* $\mathrm{P_{IV}}$ *(3.1).*

Our interest:

What more can be said about properties of polynomials orthogonal with respect to $W(x) = |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right)$?

Generalized Freud polynomials

- Let $\{S_n(x)\}_{n=0}^{\infty}$ be monic OPS related to $W(x) = |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right).$
- Three-term recurrence:

$$xS_n(x) = S_{n+1}(x) + \beta_n(t;\lambda)S_{n-1}(x);$$

where $S_{-1}(x) = 0, \quad S_0(x) = 1.$

- Symmetric property: $S_n(-x) = (-1)^n S_n(x), \quad \forall x \in \mathbb{R}.$
- The semi-classical Laguerre polynomials $L_n(w_\nu; x)$ related to $w_\nu(x) = x^\nu \exp(-x^2 + tx), \ \alpha > -1, \quad x \in \mathbb{R}^+$ via quadratic transformation generates $S_n(w_\lambda; x)$ with $W_\lambda(x) = |x|^{2\lambda+1} \exp(-x^4 + tx^2), \lambda > -1$ on $\mathbb{R}$ (Chihara, 1978; GF-WVA-LZ, 2012)

$$S_{2n}(x; W_\gamma) = L_n\left(w_{\frac{\gamma-1}{2}}; x^2\right); \quad x_{i,n}\left(w_{\frac{\gamma-1}{2}}\right) = \left[e_{i,2n}(W_\gamma)\right]^2,$$

$$S_{2n+1}(x; W_\gamma) = xL_n\left(w_{\frac{\gamma+1}{2}}; x^2\right); \quad x_{i,n}\left(w_{\frac{\gamma+1}{2}}\right) = \left[e_{i,2n+1}(W_\gamma)\right]^2.$$

Basic symmetrization principle

$\int_0^{c^2} P_m(x)\, P_n(x)\, w(x)\, dx = K_n\, \delta_{mn} \Rightarrow \int_{-c}^{c} S_m(x)\, S_n(x)\, |x|\, w(x^2)\, dx = K_n\, \delta_{mn}$, where $\rho(x) = |x|\, w(x^2)$ is a symmetric: $\rho(x) = \rho(-x)$ for all $x \in \mathbb{R}$; i.e., $\mu_{2j+1} = 0 \Leftrightarrow b_n = 0.$

Properties of generalized Freud polynomials

Moments of the generalized Freud weight [CJK, 2016]

The first moment, $\mu_0(t;\lambda)$, for generalized Freud weight in terms of the integral representation of a parabolic cylinder (Hermite-Weber) function $D_v(\xi)$:

$$\begin{aligned}\mu_0(t;\lambda) &= \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right) dx \\ &= 2\int_0^{\infty} x^{2\lambda+1} \exp\left(-x^4 + tx^2\right) dx \\ &= \int_0^{\infty} y^{\lambda} \exp\left(-y^2 + ty\right) dy \\ &= \frac{\Gamma(\lambda+1)}{2^{(\lambda+1)/2}} \exp\left(\tfrac{1}{8}t^2\right) D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right).\end{aligned}$$

since the parabolic cylinder function $D_v(\xi)$ has the integral representation

$$D_v(\xi) = \frac{\exp(-\frac{1}{4}\xi^2)}{\Gamma(-v)} \int_0^{\infty} s^{-v-1} \exp\left(-\tfrac{1}{2}s^2 - \xi s\right) ds, \qquad \Re(\nu) < 0.$$

- The even moments are

$$\begin{aligned}\mu_{2n}(t;\lambda) &= \int_{-\infty}^{\infty} x^{2n}\,|x|^{2\lambda+1} \exp\left(-x^4+tx^2\right) dx\\ &= \frac{d^n}{dt^n}\left(\int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp\left(-x^4+tx^2\right)\, dx\right),\\ &= \frac{d^n}{dt^n}\mu_0(t;\lambda), \quad n=1,2,\ldots\end{aligned}$$

whilst the odd ones are

$$\mu_{2n+1}(t;\lambda) = \int_{-\infty}^{\infty} x^{2n+1}\,|x|^{2\lambda+1} \exp\left(-x^4+tx^2\right) dx = 0, \quad n=1,2,\ldots$$

since the integrand is odd.

Differential-difference equation

Theorem (CJK, 2016)

For the generalized Freud weight $W_\lambda(x) = |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right)$, $x \in \mathbb{R}$, the monic orthogonal polynomials $S_n(x;t)$ satisfy the differential-difference equation

$$x \frac{dS_n}{dx}(x;t) = -B_n(x;t)\ S_n(x;t) + A_n(x;t)\ S_{n-1}(x;t),$$

where

$$A_n(x;t) = 4x\beta_n(x^2 - \tfrac{1}{2}t + \beta_n + \beta_{n+1}),$$

$$B_n(x;t) = 4x^2\beta_n + \frac{(2\lambda+1)[1-(-1)^n]}{2},$$

with β_n- the recurrence coefficient in the three-term recurrence relation

$$xS_n(x;t) = S_{n+1}(x;t) + \beta_n(t;\lambda)S_{n-1}(x;t).$$

Proof: Two methods:- Ladder operator method or Shohat's method.

Second order linear ODEs

Theorem (CJK, 2016)

For the generalized Freud weight

$$W_\lambda(x) = |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right), \; x \in \mathbb{R},$$

the monic orthogonal polynomials $S_n(x;t)$ satisfy the differential equation

$$\frac{d^2 S_n}{dx^2}(x;t) + R_n(x;t)\frac{dS_n}{dx}(x;t) + T_n(x;t)S_n(x;t) = 0,$$

where

$$R_n(x;t) = -4x^3 + 2tx - \frac{2\lambda+1}{x} - \frac{2x}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}},$$

$$T_n(x;t) = 4nx^2 + 4\beta_n + 16\beta_n(\beta_n + \beta_{n+1} - \tfrac{1}{2})(\beta_n + \beta_{n-1} - \tfrac{1}{2}) - \frac{8\beta_n x^2 + (2\lambda+1)[1-(-1)^n]}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}} + (2\lambda+1)[1-(-1)^n]\left(t - \frac{1}{2x^2}\right).$$

The recurrence coefficient $\beta_n(t;\lambda)$

Theorem (CJK, 2016)

The recurrence coefficients $\beta_n(t;\lambda)$ in the three-term recurrence

$$xS_n(x;t) = S_{n+1}(x;t) + \beta_n(t;\lambda)S_{n-1}(x;t)$$

where $S_{-1}(x;t) = 0$ and $S_0(x;t) = 1$ related to the weight W_λ are given by

$$\beta_{2n}(t;\lambda) = \frac{d}{dt}\ln\frac{\tau_n(t;\lambda+1)}{\tau_n(t;\lambda)}; \quad \beta_{2n+1}(t;\lambda) = \frac{d}{dt}\ln\frac{\tau_{n+1}(t;\lambda)}{\tau_n(t;\lambda+1)},$$

where $\tau_n(t;\lambda)$ is the Wronskian given by

$$\tau_n(t;\lambda) = \mathcal{W}\left(\mu_0,\mu_1,\ldots,\mu_{n-1}\right) = \mathcal{W}\left(\phi_\lambda, \frac{d\phi_\lambda}{dt},\ldots,\frac{d^{n-1}\phi_\lambda}{dt^{n-1}}\right),$$

$$\phi_\lambda(t) = \mu_0(t;\lambda) = \frac{\Gamma(\lambda+1)}{2^{(\lambda+1)/2}}\exp\left(\tfrac{1}{8}t^2\right)D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right).$$

with $D_v(\xi)$, with $v \notin \mathbb{Z}$, is the parabolic cylinder function.

Sample recurrence coefficients in terms of Φ_λ

The first few recurrence coefficients $\beta_n(t;\lambda)$ are given by

$$
\begin{aligned}
\beta_1(t;\lambda) &= \Phi_\lambda,\\
\beta_2(t;\lambda) &= -\frac{2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1}{2\Phi_\lambda},\\
\beta_3(t;\lambda) &= -\frac{\Phi_\lambda}{2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1} - \frac{\lambda+1}{2\Phi_\lambda},\\
\beta_4(t;\lambda) &= \frac{t}{2(\lambda+2)} + \frac{\Phi_\lambda}{2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1}\\
&\qquad + \frac{(\lambda+1)(t^2+2\lambda+4)\Phi_\lambda + (\lambda+1)^2 t}{2(\lambda+2)[2(\lambda+2)\Phi_\lambda^2 - (\lambda+1)t\Phi_\lambda - (\lambda+1)^2]},
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_\lambda(t) &= \frac{d}{dt}\ln\left\{D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)\exp\left(\tfrac{1}{8}t^2\right)\right\}\\
&= \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\,\frac{D_{-\lambda}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)}{D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)}.
\end{aligned}
$$

Some of the polynomials

By using the recurrence $xS_n(x;t) = S_{n+1}(x;t) + \beta_n(t;\lambda)S_{n-1}(x;t)$, the first few polynomials:

$$S_1(x;t,\lambda) = x,$$

$$S_2(x;t,\lambda) = x^2 - \Phi_\lambda,$$

$$S_3(x;t,\lambda) = x^3 + \frac{t\Phi_\lambda + \lambda + 1}{2\Phi_\lambda}\,x,$$

$$S_4(x;t,\lambda) = x^4 + \frac{2t\Phi_\lambda^2 - (t^2+2)\Phi_\lambda - (\lambda+1)t}{2(2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1)}\,x^2 - \frac{2(\lambda+2)\Phi_\lambda^2 - (\lambda+1)t\Phi_\lambda - (\lambda+1)^2}{2(2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1)},$$

$$S_5(x;t,\lambda) = x^5 - \frac{2(\lambda+3)t\Phi_\lambda^2 - (\lambda+1)(t^2-2)\Phi_\lambda - (\lambda+1)^2 t}{4(\lambda+2)\Phi_\lambda^2 - 2(\lambda+1)t\Phi_\lambda - 2(\lambda+1)^2}\,x^3 - \frac{\left[2(\lambda+2)^2 - t^2\right]\Phi_\lambda^2 - (\lambda+1)(\lambda+4)t\Phi_\lambda - (\lambda+1)^2(\lambda+3)}{4(\lambda+2)\Phi_\lambda^2 - 2(\lambda+1)t\Phi_\lambda - 2(\lambda+1)^2}\,x.$$

Thank you very much

for your

kind attention!

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