Characterizing properties of generalized Freud polynomials

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Outline

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§18.32 OP’s with Respect to Freud Weights

A Freud weight is a weight function of the form

\[ w(x) = \exp(-Q(x)), \quad -\infty < x < \infty \]

where \( Q(x) \) is real, even, non-negative, and continuously differentiable. Of special interest are the cases \( Q(x) = x^{2m}, \quad m = 1, 2, \ldots \). No explicit expressions for the corresponding OP’s are available. However, for asymptotic approximations in terms of elementary functions for the OP’s, and also for their largest zeros, see Levin and Lubinsky [2001] and Nevai [1986]. For a uniform asymptotic expansion in terms of Airy functions for the OP’s in the case \( x^4 \) see Bo and Wong [1999].
Orthogonal polynomial sequences

- Given \( \{\mu_n\} \in \mathbb{R} \), we define the moment functional \( \mathcal{L} : x^n \to \mu_n \) on the linear space of polynomials \( \mathcal{P} \). Assume \( \mu_0 = \mathcal{L}(1) = 1 \). The inner product \( \langle \cdot, \cdot \rangle \) for the functional \( \mathcal{L} \) is given by

  \[
  \langle P_m(x), P_n(x) \rangle = \mathcal{L}(P_m(x)P_n(x))
  \]

- Monic polynomials \( \{P_n(x)\}_{n=0}^{\infty} \) orthogonal w.r.t. a moment functional \( \mathcal{L} \) related to an absolutely continuous Borel measure \( \mu \) on \( \mathbb{R} \); \( d\mu(x) = w(x) \, dx \); \( w(x) > 0 \):

  \[
  \mathcal{L}(P_m(x)P_n(x)) = \langle P_m, P_n \rangle = \int_{\mathbb{R}} P_m(x)P_n(x) \, d\mu(x) = h_n \, \delta_{mn},
  \]

  where the normalization constant \( h_n > 0 \) and \( \delta_{mn} \) is the Kronecker delta.
Monic orthogonal polynomials \( P_n(x) \) satisfy

\[
P_{-1}(x) = 0, \quad P_0(x) = 1, \quad P_{n+1}(x) = (x - \alpha_n) P_n(x) - \beta_n P_{n-1}(x),
\]

\[
\alpha_n = \frac{\langle xP_n, P_n \rangle}{\langle P_n, P_n \rangle} \in \mathbb{R}; \quad \beta_n = \frac{\langle P_n, P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle} > 0, \quad \beta_0 = 1, \quad n \in \mathbb{N}_0,
\]

and the constant:

\[
h_n = \langle P_n, P_n \rangle = \| P_n \|^2 = \prod_{j=1}^{n} \beta_j.
\]

To construct \( P_n(x) \) for \( \mathcal{L} \):

\[
P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix}
\mu_0 & \mu_1 & \cdots & \mu_n \\
\mu_1 & \mu_2 & \cdots & \mu_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\
1 & x & \cdots & x^n
\end{vmatrix}, \quad \Delta_n := \det(\mu_{i+j})_{i,j=0}^n > 0.
\]
Classical orthogonal polynomials

- Classical weights satisfy **Pearson's equation**
  \[
  \frac{d}{dx} (\sigma w) = \tau w, \tag{2.1}
  \]
  with \(\text{deg}(\sigma) \leq 2\) and \(\text{deg}(\tau) = 1\), and bcs: \(\sigma(x) \ w(x) = 0\) for \(x = a\) and \(x = b\).

<table>
<thead>
<tr>
<th>(p_n)</th>
<th>(w(x))</th>
<th>(\sigma(x))</th>
<th>(\tau(x))</th>
<th>interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hermite</td>
<td>(\exp(-x^2))</td>
<td>1</td>
<td>(-2x)</td>
<td>((-\infty, \infty))</td>
</tr>
<tr>
<td>Laguerre</td>
<td>(x^\alpha \exp(-x), \ \alpha &gt; -1)</td>
<td>(x)</td>
<td>(1 + \alpha - x)</td>
<td>((0, \infty))</td>
</tr>
<tr>
<td>Jacobi</td>
<td>((1 - x)^\alpha (1 + x)^\beta)</td>
<td>(1 - x^2)</td>
<td>(\beta - \alpha - (2 + \alpha + \beta)x)</td>
<td>([-1, 1])</td>
</tr>
</tbody>
</table>

- \(p_n\)'s are solutions of \(Lp_n = \lambda_n p_n\) where \(L\) is a second order differential operator (Sturm-Liouville) [Bochner, 1929]
- **Structural relation:**
  \[
  \sigma(x) (p_n(x))' = \sum_{j=n-1}^{n-r+1} A_{n,j} p_j(x), \quad r = \text{deg}(\sigma) \tag{2.2}
  \]

- (2.2) together with \(xp_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}\), yields a first order recurrence equation for the recurrence coefficients \(a_n\) and \(b_n\), which can be solved explicitly.
Semi-classical orthogonal polynomials

- **Semi-classical weights** satisfy Pearson’s equation (2.1) with \( \deg(\sigma) > 2 \) or \( \deg(\tau) > 1 \). [Hendriksen, van Rossum, 1977]

<table>
<thead>
<tr>
<th>weight</th>
<th>( w(x) )</th>
<th>parameters</th>
<th>( \sigma(x) )</th>
<th>( \tau(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>( \exp(-x^4) )</td>
<td>-</td>
<td>1</td>
<td>(-4x^3)</td>
</tr>
<tr>
<td>Airy</td>
<td>( \exp(-\frac{1}{3}x^3 + tx) )</td>
<td>( t &gt; 0 )</td>
<td>1</td>
<td>( t - x^2 )</td>
</tr>
<tr>
<td>Semi-classical Laguerre</td>
<td>( x^\lambda \exp(-x^2 + tx) )</td>
<td>( \lambda &gt; -1 )</td>
<td>( x )</td>
<td>1 + ( \lambda + tx - 2x^2 )</td>
</tr>
<tr>
<td>Freud</td>
<td>( \exp(-\frac{1}{4}x^4 - tx^2) )</td>
<td>( x, t \in \mathbb{R} )</td>
<td>1</td>
<td>(-2tx - x^3)</td>
</tr>
<tr>
<td>Generalized Freud</td>
<td>(</td>
<td>x</td>
<td>^{2\lambda+1}\exp(-x^4 + tx^2) )</td>
<td>( \lambda &gt; 0 ), ( x, t \in \mathbb{R} )</td>
</tr>
</tbody>
</table>

- \( p_n \) does not satisfy Sturm-Liouville differential equation.

- **Structural relation**

\[
\sigma(x)p_n'(x) = \sum_{j=n-s}^{n-r+1} A_{n,j}p_j(x), \quad \begin{cases} r = \deg(\sigma), \\ s = \max\{\deg(\sigma) - 1, \deg(\tau)\} \end{cases} \tag{2.3}
\]

(2.3) and \( xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \ n \geq 0 \), yield second or higher order (non-linear) equations for the recurrence coefficients \( a_n \) and \( b_n \).

Example: \( w(x) = \exp(-x^4) \) on \( \mathbb{R} \) [Nevai, 1983]: \( b_n = 0 \) (symmetry);

\[
4a_n^2 \left( a_n^2 + a_n^2 + a_n^2 \right) = n, \ n \geq 2, \ a_0 = 1, \ a_1^2 = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}, \text{ where } \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.
\]
The link to Painlevé equations

- **Some history**: The first non-linear recurrence equation – Shohat (1930’s) and Laguerre, Freud (late 70’s) and very recently recognized as discrete Painlevé equations by Fokas, Its, and Kitaev. Work by Magnus (relation between discrete and continuous Painlevé equations), Witte, Clarkson, Van Assche, Nijhoff, Spicer, Chen and Ismail extended theory with some more examples.

- **Some Discrete Painlevé eqns**:
  
  \[(d-\text{P}_I) \quad x_{n+1} + x_n + x_{n-1} = \frac{z_n + \gamma(-1)^n}{x_n} + \sigma\]

  \[(d-\text{P}_{II}) \quad x_{n+1} + x_{n-1} = \frac{x_n z_n + \gamma}{1 - x_n^2}\]

  \[(d-\text{P}_{IV}) \quad (x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - \kappa^2)(x_n^2 - \mu^2)}{(x_n + z_n)^2 - \gamma^2}\]

- The continuous fourth Painlevé equation (P_{IV})

  \[
  \frac{d^2 q}{dz^2} = \frac{1}{2q} \left( \frac{dq}{dz} \right)^2 + \frac{3}{2} q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q}, \tag{3.1}
  \]

  where \(A\) and \(B\) are constants, which are expressed in terms of parabolic cylinder (Hermite-Weber) functions.
Semi-classical Laguerre

Theorem (LB-WVA, 2012)

The coefficients $\alpha_n(t)$ and $\beta_n(t)$ in the three-term recurrence

$$L_{n+1}^{(\nu)}(x; t) = (x - \alpha_n)L_n^{(\nu)}(x; t) - \beta_nL_{n-1}^{(\nu)}(x; t);$$

associated with the semi-classical Laguerre $w_\nu(x) = x^\nu \exp(-x^2 + tx)$, $\nu > -1$, $x \in \mathbb{R}^+$ are:

$$(2\alpha_n - t)(2\alpha_{n-1} - t) = \frac{(2\beta_n - n)(2\beta_n - n - \nu)}{\beta_n},$$

$$2\beta_n + 2\beta_{n+1} - \alpha_n(2\alpha_n - t) = 2n + 1 + \nu.$$

For explicit formulations of $\alpha_n$ and $\beta_n$, see [CJ, 2014].
**Question**: What semi-classical weights are related to discrete Painlevé equations? Which discrete Painlevé equations do we obtain?

- \( w(x) = |x|^\varrho \exp(-x^4), \ \varrho > -1 \) on \( \mathbb{R} \) is related to (d-P\(_I\)). [Magnus, 1986].
- \( w(x) = x^\alpha \exp(-x^2), \ \alpha > -1 \) on \( \mathbb{R}^+ \) is related to (d-P\(_{IV}\)) [Sonin-type].
- \( w(x; t) = x^\alpha \exp(-x^2 + tx), \ \alpha > -1 \) on \( \mathbb{R}^+ \) is related to (P\(_{IV}\)) [GF-WVA-LZ, 2011].
- \( W_\lambda(x) = |x|^{2\lambda+1} \exp(-x^4 + tx^2), \ \lambda > -1, \ t, x \in \mathbb{R} \) related to (d-P\(_I\)) and continuous (P\(_{IV}\)) [LB-WVA, 2011, GF-WVA-LZ, 2012].
The link to Painlevé equations

The recurrence coefficient related to Painlevé IV

Theorem. (LB-WVA, 2011; GF-WVA-LZ, 2012)

The recurrence coefficients $\beta_n(t; \lambda)$ in the three term recurrence

$$xS_n(x; t) = S_{n+1}(x; t) + \beta_n(t; \lambda)S_{n-1}(x; t)$$

associated with the weight $W_\lambda$ satisfy the equation

$$\frac{d^2\beta_n}{dt^2} = \frac{1}{2\beta_n} \left( \frac{d\beta_n}{dt} \right)^2 + \frac{3}{2} \beta_n^3 - t\beta_n^2 + \left( \frac{1}{8} t^2 - \frac{1}{2} A_n \right) \beta_n + \frac{B_n}{16\beta_n}, \quad (3.2)$$

where the parameters $A_n$ and $B_n$ are given by

$$A_{2n} = -2\lambda - n - 1, \quad A_{2n+1} = \lambda - n,$$
$$B_{2n} = -2n^2, \quad B_{2n+1} = -2(\lambda + n + 1)^2.$$

Further $\beta_n(t)$ satisfies the non-linear difference equation

$$\beta_{n+1} + \beta_n + \beta_{n-1} = \frac{1}{2} t + \frac{2n + (2\lambda + 1)[1 - (-1)^n]}{8\beta_n}, \quad \text{discrete } P_1 \ (dP_1).$$

Remark: $(3.2) \equiv P_{IV}$ via the transformation $\beta_n(t; \lambda) = \frac{1}{2} w(z), \text{ with } z = -\frac{1}{2} t.$ Hence

$$\beta_{2n}(t; \lambda) = \frac{1}{2} w(z; -2\lambda - n - 1, -2n^2); \quad \beta_{2n+1}(t; \lambda) = \frac{1}{2} w(z; \lambda - n, -2(\lambda + n + 1)^2),$$

with $z = -\frac{1}{2} t,$ where $w(z; A, B)$ satisfies $P_{IV}$ ($3.1$).
Our interest:

What more can be said about properties of polynomials orthogonal with respect to $W(x) = |x|^{2\lambda+1} \exp(-x^4 + tx^2)$?
Generalized Freud polynomials

- Let \( \{S_n(x)\}_{n=0}^{\infty} \) be monic OPS related to \( W(x) = |x|^{2\lambda+1} \exp(-x^4 + tx^2) \).

- Three-term recurrence:
  \[
xS_n(x) = S_{n+1}(x) + \beta_n(t; \lambda) S_{n-1}(x);
\]
  where \( S_{-1}(x) = 0, \ S_0(x) = 1. \)

- Symmetric property: \( S_n(-x) = (-1)^n S_n(x), \ \forall x \in \mathbb{R}. \)

- The semi-classical Laguerre polynomials \( L_n(w_\nu; x) \) related to \( w_\nu(x) = x^\nu \exp(-x^2 + tx), \ \alpha > -1, \ x \in \mathbb{R}^+ \) via quadratic transformation generates \( S_n(w_\lambda; x) \) with \( W_\lambda(x) = |x|^{2\lambda+1} \exp(-x^4 + tx^2), \lambda > -1 \) on \( \mathbb{R} \) (Chihara, 1978; GF-WVA-LZ, 2012)
  \[
  S_{2n}(x; W_\gamma) = L_n \left( \frac{w_{\gamma-1}}{2}; x^2 \right); \quad x_{i,n} \left( \frac{w_{\gamma-1}}{2} \right) = [e_{i,2n}(W_\gamma)]^2, \\
  S_{2n+1}(x; W_\gamma) = xL_n \left( \frac{w_{\gamma+1}}{2}; x^2 \right); \quad x_{i,n} \left( \frac{w_{\gamma+1}}{2} \right) = [e_{i,2n+1}(W_\gamma)]^2.
  \]

- Basic symmetrization principle
  \[
  \int_{c^2} P_m(x) P_n(x) w(x) \, dx = K_n \delta_{mn} \Rightarrow \int_{-c}^{c} S_m(x) S_n(x) |x| w(x^2) \, dx = K_n \delta_{mn}, \text{ where } \rho(x) = |x| w(x^2)
  \]
  is a symmetric: \( \rho(x) = \rho(-x) \) for all \( x \in \mathbb{R}; \ i.e., \mu_{2j+1} = 0 \Leftrightarrow b_n = 0. \)
Properties of generalized Freud polynomials

Moments of the generalized Freud weight [CJK, 2016]

The first moment, $\mu_0(t; \lambda)$, for generalized Freud weight in terms of the integral representation of a parabolic cylinder (Hermite-Weber) function $D_\nu(\xi)$:

\[
\mu_0(t; \lambda) = \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(-x^4 + tx^2) \, dx
\]
\[
= 2 \int_{0}^{\infty} x^{2\lambda+1} \exp(-x^4 + tx^2) \, dx
\]
\[
= \int_{0}^{\infty} y^{\lambda} \exp(-y^2 + ty) \, dy
\]
\[
= \frac{\Gamma(\lambda + 1)}{2^{(\lambda+1)/2}} \exp\left(\frac{1}{8} t^2\right) D_{-\lambda-1}\left(-\frac{1}{2} \sqrt{2} t\right).
\]

since the parabolic cylinder function $D_\nu(\xi)$ has the integral representation

\[
D_\nu(\xi) = \frac{\exp\left(-\frac{1}{4} \xi^2\right)}{\Gamma(-\nu)} \int_{0}^{\infty} s^{-\nu-1} \exp\left(-\frac{1}{2} s^2 - \xi s\right) \, ds, \quad \Re(\nu) < 0.
\]
The even moments are

\[ \mu_{2n}(t; \lambda) = \int_{-\infty}^{\infty} x^{2n} |x|^{2\lambda+1} \exp(-x^4 + tx^2) \, dx \]

\[ = \frac{d^n}{dt^n} \left( \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(-x^4 + tx^2) \, dx \right), \]

\[ = \frac{d^n}{dt^n} \mu_0(t; \lambda), \quad n = 1, 2, \ldots \]

whilst the odd ones are

\[ \mu_{2n+1}(t; \lambda) = \int_{-\infty}^{\infty} x^{2n+1} |x|^{2\lambda+1} \exp(-x^4 + tx^2) \, dx = 0, \quad n = 1, 2, \ldots \]

since the integrand is odd.
Theorem (CJK, 2016)

For the generalized Freud weight \( W_\lambda(x) = |x|^{2\lambda+1} \exp(-x^4 + tx^2) \), \( x \in \mathbb{R} \), the monic orthogonal polynomials \( S_n(x; t) \) satisfy the differential-difference equation

\[
 x \frac{d S_n}{d x}(x; t) = -B_n(x; t) S_n(x; t) + A_n(x; t) S_{n-1}(x; t),
\]

where

\[
 A_n(x; t) = 4x\beta_n(x^2 - \frac{1}{2} t + \beta_n + \beta_{n+1}),
\]

\[
 B_n(x; t) = 4x^2\beta_n + \frac{(2\lambda + 1)[1 - (-1)^n]}{2},
\]

with \( \beta_n \) - the recurrence coefficient in the three-term recurrence relation

\[
 xS_n(x; t) = S_{n+1}(x; t) + \beta_n(t; \lambda)S_{n-1}(x; t).
\]

Proof: Two methods:– Ladder operator method or Shohat’s method.
Theorem (CJK, 2016)

For the generalized Freud weight

\[ W_\lambda(x) = |x|^{2\lambda+1} \exp \left( -x^4 + tx^2 \right), \ x \in \mathbb{R}, \]

the monic orthogonal polynomials \( S_n(x; t) \) satisfy the differential equation

\[ \frac{d^2 S_n}{dx^2}(x; t) + R_n(x; t) \frac{d S_n}{dx}(x; t) + T_n(x; t)S_n(x; t) = 0, \]

where

\[ R_n(x; t) = -4x^3 + 2tx - \frac{2\lambda + 1}{x} - \frac{2x}{x^2 - \frac{1}{2} t + \beta_n + \beta_{n+1}}, \]

\[ T_n(x; t) = 4nx^2 + 4\beta_n + 16\beta_n(\beta_n + \beta_{n+1} - \frac{1}{2})(\beta_n + \beta_{n-1} - \frac{1}{2}) \]

\[ - \frac{8\beta_n x^2 + (2\lambda + 1)[1 - (-1)^n]}{x^2 - \frac{1}{2} t + \beta_n + \beta_{n+1}} + (2\lambda + 1)[1 - (-1)^n] \left( t - \frac{1}{2x^2} \right). \]
The recurrence coefficient $\beta_n(t; \lambda)$

**Theorem (CJK, 2016)**

The recurrence coefficients $\beta_n(t; \lambda)$ in the three-term recurrence

$$xS_n(x; t) = S_{n+1}(x; t) + \beta_n(t; \lambda)S_{n-1}(x; t)$$

where $S_{-1}(x; t) = 0$ and $S_0(x; t) = 1$ related to the weight $W_\lambda$ are given by

$$\beta_{2n}(t; \lambda) = \frac{d}{dt} \ln \frac{\tau_n(t; \lambda + 1)}{\tau_n(t; \lambda)}; \quad \beta_{2n+1}(t; \lambda) = \frac{d}{dt} \ln \frac{\tau_{n+1}(t; \lambda)}{\tau_n(t; \lambda + 1)},$$

where $\tau_n(t; \lambda)$ is the Wronskian given by

$$\tau_n(t; \lambda) = \mathcal{W}(\mu_0, \mu_1, \ldots, \mu_{n-1}) = \mathcal{W} \left( \phi_\lambda, \frac{d\phi_\lambda}{dt}, \ldots, \frac{d^{n-1}\phi_\lambda}{dt^{n-1}} \right),$$

$$\phi_\lambda(t) = \mu_0(t; \lambda) = \frac{\Gamma(\lambda + 1)}{2^{(\lambda+1)/2}} \exp \left( \frac{1}{8} t^2 \right) D_{-\lambda-1} \left( -\frac{1}{2} \sqrt{2} t \right).$$

with $D_\nu(\xi)$, with $\nu \notin \mathbb{Z}$, is the parabolic cylinder function.
Sample recurrence coefficients in terms of $\Phi_{\lambda}$

The first few recurrence coefficients $\beta_n(t; \lambda)$ are given by

\[
\begin{align*}
\beta_1(t; \lambda) &= \Phi_{\lambda}, \\
\beta_2(t; \lambda) &= -\frac{2\Phi_{\lambda}^2 - t\Phi_{\lambda} - \lambda - 1}{2\Phi_{\lambda}}, \\
\beta_3(t; \lambda) &= -\frac{\Phi_{\lambda}}{2\Phi_{\lambda}^2 - t\Phi_{\lambda} - \lambda - 1} - \frac{\lambda + 1}{2\Phi_{\lambda}}, \\
\beta_4(t; \lambda) &= \frac{t}{2(\lambda + 2)} + \frac{\Phi_{\lambda}}{2\Phi_{\lambda}^2 - t\Phi_{\lambda} - \lambda - 1} \\
&\quad + \frac{(\lambda + 1)(t^2 + 2\lambda + 4)\Phi_{\lambda} + (\lambda + 1)^2 t}{2(\lambda + 2)[2(\lambda + 2)\Phi_{\lambda}^2 - (\lambda + 1)t\Phi_{\lambda} - (\lambda + 1)^2]},
\end{align*}
\]

where

\[
\Phi_{\lambda}(t) = \frac{d}{dt} \ln \left\{ D_{-\lambda - 1}( -\frac{1}{2} \sqrt{2} t) \exp \left( \frac{1}{8} t^2 \right) \right\}
= \frac{1}{2} t + \frac{1}{2} \sqrt{2} \frac{D_{-\lambda}( -\frac{1}{2} \sqrt{2} t)}{D_{-\lambda - 1}( -\frac{1}{2} \sqrt{2} t)}.
\]
Some of the polynomials

By using the recurrence \( xS_n(x; t) = S_{n+1}(x; t) + \beta_n(t; \lambda)S_{n-1}(x; t) \), the first few polynomials:

\[
S_1(x; t, \lambda) = x,
\]
\[
S_2(x; t, \lambda) = x^2 - \Phi_\lambda,
\]
\[
S_3(x; t, \lambda) = x^3 + \frac{t\Phi_\lambda + \lambda + 1}{2\Phi_\lambda} x,
\]
\[
S_4(x; t, \lambda) = x^4 + \frac{2t\Phi_\lambda^2 - (t^2 + 2)\Phi_\lambda - (\lambda + 1)t}{2(2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1)} x^2
\]
\[
- \frac{2(\lambda + 2)\Phi_\lambda^2 - (\lambda + 1)t\Phi_\lambda - (\lambda + 1)^2}{2(2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1)},
\]
\[
S_5(x; t, \lambda) = x^5 - \frac{2(\lambda + 3)t\Phi_\lambda^2 - (\lambda + 1)(t^2 - 2)\Phi_\lambda - (\lambda + 1)^2 t}{4(\lambda + 2)\Phi_\lambda^2 - 2(\lambda + 1)t\Phi_\lambda - 2(\lambda + 1)^2} x^3
\]
\[
- \frac{[2(\lambda + 2)^2 - t^2]\Phi_\lambda^2 - (\lambda + 1)(\lambda + 4)t\Phi_\lambda - (\lambda + 1)^2(\lambda + 3)}{4(\lambda + 2)\Phi_\lambda^2 - 2(\lambda + 1)t\Phi_\lambda - 2(\lambda + 1)^2} x.
\]
Thank you very much for your kind attention!
References

[C, 2010] PA Clarkson, Recurrence coefficients for discrete orthonormal polynomials and the Painlevé equations, School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury, CT2 7NF, UK.


