Hyperbolic hydraulic fracture with tortuosity

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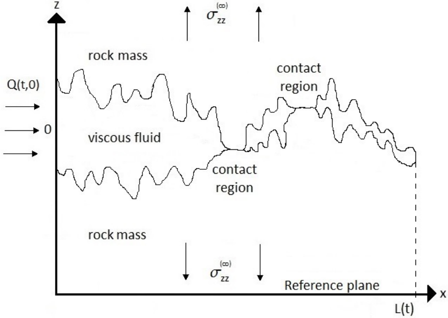
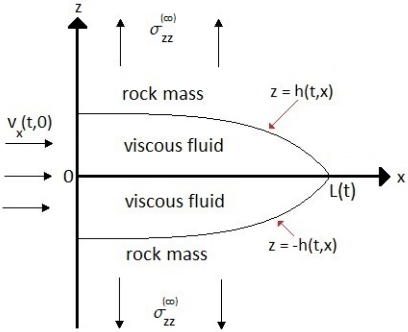
Model formulation

Problem description

(a) Tortuous fracture



(b) Two-dimensional symmetric model



$$v_x = v_x(x,z,t), \quad v_y = 0, \quad v_z = v_z(x,z,t), \quad p = p(x,z,t),$$

- Reynolds flow law
- General flow law (Fitt et al): Substitute h^3 by $a_n h^n$

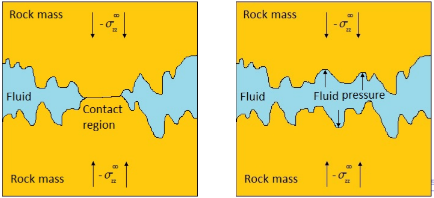
  * Fluid flux: $Q(t,x) = -\dfrac{2}{3\mu} a_n h^n \dfrac{\partial p}{\partial x}(t,x),$

  * Width averaged fluid velocity: $\overline{v}_x(t,x) = -\dfrac{a_n}{3\mu} h^{n-1} \dfrac{\partial p}{\partial x}(t,x),$

  * Governing PDE: $\dfrac{\partial h}{\partial t} = \dfrac{a_n}{3\mu}\dfrac{\partial}{\partial x}\left(h^n \dfrac{\partial p}{\partial x}\right).$

- Crack laws

(a) Partially open fracture (b) Open fracture



- PKN approximation: $\sigma_{zz}(t,x) = \sigma_{zz}^{(\infty)} - \Lambda h(t,x), \quad \Lambda = \dfrac{E}{(1-\nu^2)B}$
- Linear crack law (Pine et al [3], Fitt et al [1], Kgatle & Mason [2])
- Hyperbolic crack law (Goodman [4])

$$p(t,x) + p_2(t,x) = -\sigma_{zz}(t,x), \quad p_2(t,x) = -k\left(\frac{h_{max} - h(t,x)}{h(t,x) - h_{min}}\right)$$

where $k < 0$.

* $h_{min} << h_{max}$, $\therefore$ assume $h_{min} = 0$ (Fitt et al [1], King and Please [9])
* Pressure gradient: $\dfrac{\partial p}{\partial x}(t,x) = \left(\Lambda - \dfrac{k h_{max}}{h^2}\right)\dfrac{\partial h}{\partial x}.$
* Transformation variables: $x^* = \dfrac{x}{L_o}, \quad h^* = \dfrac{h}{h_{max}}, \quad t^* = \dfrac{Ut}{L_o},$

$$L^* = \frac{L}{L_o}, \quad \overline{v}_x^* = \frac{\overline{v}_x}{U}, \quad Q^* = \frac{Q}{h_{max}U}, \quad U = \frac{\Lambda h_{max}^3}{\mu L_o}\left(1 - \frac{\sigma_{zz}^{(\infty)}}{\Lambda h_{max}}\right)$$

Governing equations

* Governing PDE: $\dfrac{\partial h^*}{\partial t^*} = K_n \dfrac{\partial}{\partial x^*}\left({h^*}^n \dfrac{\partial h^*}{\partial x^*} + \phi {h^*}^{n-2} \dfrac{\partial h^*}{\partial x^*} \right)$
* BCs:

$$h^*(t^*, L(t)) = 0,$$

$$-2K_n\left({h^*}^n(t,0)\frac{\partial h^*}{\partial x^*}(t^*,0) + \phi {h^*}^{n-2}(t^*,0)\frac{\partial h^*}{\partial x^*}(t^*,0)\right) = \frac{dV^*}{dt^*},$$

* Fluid flux: $Q^*(t^*, x^*) = -2K_n\left({h^*}^n \dfrac{\partial h^*}{\partial x^*} + \phi {h^*}^{n-2} \dfrac{\partial h^*}{\partial x^*}\right),$
* Width averaged velocity: $\overline{v}^*_x(t^*, x^*) = -K_n\left({h^*}^{n-1} \dfrac{\partial h^*}{\partial x^*} + \phi {h^*}^{n-3} \dfrac{\partial h^*}{\partial x^*}\right)$

$$K_n = \frac{a_n h_{max}^{n-3}}{3}\left(\frac{1}{1-\frac{\sigma_{zz}^{(\infty)}}{\Lambda h_{max}}}\right), \quad \phi = -\frac{k}{\Lambda h_{max}}, \quad k < 0.$$

Group invariant solution

$$\frac{\partial h}{\partial t} = K_n \frac{\partial}{\partial x}\left(h^n \frac{\partial h}{\partial x} + \phi h^{n-2} \frac{\partial h}{\partial x} \right).$$

- Other methods of solution (Huppert [7], Spence and Sharp [10])
- Lie point symmetry generator:

$$\mathbf{X} = (c_1 + c_2 t)\frac{\partial}{\partial t} + (c_3 + \frac{c_2}{2} x)\frac{\partial}{\partial x}$$

where c_1, c_2, and c_3 are constants.

∗ $X(\phi - h)|_{\phi=h} = 0 \rightarrow$ a linear PDE $\rightarrow$ Group invariant solution:

$$h = F(\xi) = \left[\left(\frac{c_2}{c_1}\right) \frac{1}{2K_n} \right]^{\frac{1}{n}} f(u), \quad \xi = \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} u, \quad u = \frac{x}{L(t)}.$$

∗ Important half-width condition: $h^*(0,0) = \dfrac{h(0,0)}{h_{max}} = \beta$

∗ Partially open fracture: $0 \leq \beta < 1$

The problem is to solve

* BVP:

$$\frac{d}{du}\left(f^n\frac{df}{du}+\frac{\phi f^2(0)}{\beta^2}f^{n-2}\frac{df}{du}\right)+\frac{d}{du}(uf)-f=0,$$

$$f(1)=0,$$

$$f^n(0)\frac{df}{du}(0)=-\frac{1}{\left(1+\frac{\phi}{\beta^2}\right)}\int_0^1 f(u)du.$$

* Length: $L(t)=\left(1+2\left(\frac{\beta}{f(0)}\right)^n K_n t\right)^{\frac{1}{2}},$

* Volume: $V(t)=2\beta L(t)\int_0^1\frac{f(u)}{f(0)}du,$

* Half-width: $h(t,x)=\beta\frac{f(u)}{f(0)}.$

* Fluid flux:

$$Q(t,x) = -2\frac{K_n}{L(t)}\left(\frac{\beta}{f(0)}\right)^{n+1}\left(f^n + \frac{\phi f^2(0)}{\beta^2} f^{n-2}\right)\frac{\partial f}{\partial u},$$

* Width averaged velocity:

$$\overline{v}_x(t,x) = -\frac{K_n}{L(t)}\left(\frac{\beta^n}{f^n(0)}\right)\left(f^{n-1} + \frac{\phi f^2(0)}{\beta^2} f^{n-3}\right)\frac{\partial f}{\partial u},$$

where

$$K_n = \frac{a_n h_{max}^{n-3}}{3}\left(\frac{1}{1-\frac{\sigma_{zz}^{(\infty)}}{\Lambda h_{max}}}\right), \quad \phi = -\frac{k}{\Lambda h_{max}}$$

and

$$0 \leq \beta < 1.$$

Operating conditions

Conservation laws

- Double reduction theorem (Sjöberg [8])
- Conservation law for a PDE

$$D_t T^1 + D_x T^2 \Big|_{PDE} = 0$$

where D_t and D_x are total derivatives

$$D_t = \frac{\partial}{\partial t} + h_t \frac{\partial}{\partial h} + h_{tt} \frac{\partial}{\partial h_t} + h_{xt} \frac{\partial}{\partial h_x} + \dots$$

$$D_x = \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h} + h_{tx} \frac{\partial}{\partial h_t} + h_{xx} \frac{\partial}{\partial h_x} + \dots$$

respectively and $\mathbf{T} = (T^1, T^2)$ is a conserved vector.

- New conserved vector (Kara and Mahomed [12]):

$$\mathbf{T}^* = X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i), \quad i = 1, 2.$$

- Association (Kara and Mahomed [13]): $\mathbf{T}^* = 0.$

- Hyperbolic hydraulic fracture with tortuosity

$$\frac{\partial h}{\partial t} = K_n \frac{\partial}{\partial x}\left(h^n \frac{\partial h}{\partial x} + \phi h^{n-2}\frac{\partial h}{\partial x}\right)$$

∗ From the elementary conservation law, the conserved vector is

$\mathbf{T}_{(1)} = (h, -K_n(h^n + \phi h^{n-2})h_x),$

New conserved vector: $\mathbf{T}^*_{(1)} = \frac{c_2}{2}\mathbf{T}_{(1)}.$

∗ From the second conservation law, the conserved vector is

$$\mathbf{T}_{(2)} = \left(xh, K_n\left[\frac{h^{n+1}}{(n+1)} + \phi\frac{h^{n-1}}{(n-1)} - x(h^n + \phi h^{n-2})h_x\right]\right),$$

New conserved vector: $\mathbf{T}^*_{(2)} = c_3\mathbf{T}_{(1)} + c_2\mathbf{T}_{(2)}.$

∗ Association for non-trivial solutions is not satisfied

Comparison of Lie point symmetries

- Hyperbolic hydraulic fracture:

$$X = (c_1 + c_2 t)\frac{\partial}{\partial t} + (c_3 + \frac{c_2}{2}x)\frac{\partial}{\partial x}$$

- Linear hydraulic fracture:

$$X = (c_1 + c_2 t)\frac{\partial}{\partial t} + (c_3 + c_4 x)\frac{\partial}{\partial x} + \frac{1}{n}(2c_4 - c_2)h\frac{\partial}{\partial h}$$

$$X = \left(\frac{c_1}{c_2} + t\right)\frac{\partial}{\partial t} + \left(\frac{c_3}{c_2} + \alpha x\right)\frac{\partial}{\partial x} + \frac{1}{n}(2\alpha - 1)h\frac{\partial}{\partial h}$$

$$\eta = \frac{1}{n}(2\alpha - 1)h = 0, \quad \text{provided} \quad \alpha = \frac{1}{2}$$

$$X = (c_1 + c_2 t)\frac{\partial}{\partial t} + (c_3 + \frac{c_2}{2}x)\frac{\partial}{\partial x}$$

- Constant pressure working condition

Numerical solution

Method of solution

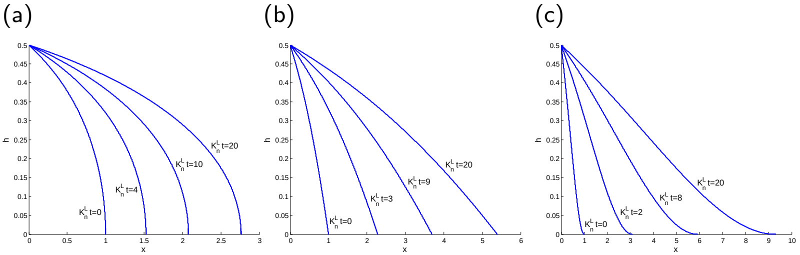
- BVP $\to$ 2 IVPs
  - $*$ Transformation variables: $\bar{u} = \gamma u, \quad \bar{f} = \gamma^{-\frac{2}{n}} f.$
- Asymptotic solution, (as $u \to 1$)

$$f(u) \sim \left[(n-2)\frac{\beta^2}{\phi f^2(0)}\right]^{\frac{1}{n-2}} (1-u)^{\frac{1}{n-2}}, \quad \text{for} \quad 2 < n < 5,$$

$$h(t,x) \sim \beta \left[(n-2)\frac{\beta^2}{\phi f^2(0)}\right]^{\frac{1}{n-2}} \left(1 - \frac{x}{L(t)}\right)^{\frac{1}{n-2}},$$

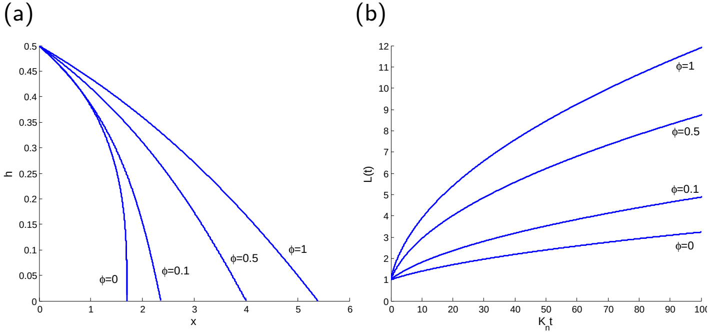
$$\frac{\partial h}{\partial x}(t, L(t)) \sim \begin{cases} -\infty, & n > 3 \\ -\frac{1}{\phi L(t)}\left(\frac{\beta}{f(0)}\right)^3, & n = 3 \\ 0, & 2 \leq n < 3 \end{cases}$$

Numerical results



Partially open fracture ($\beta = 0.5$) propagating with fluid injected at the fracture entry at a constant pressure. The numerical solution for the half-width $h(t, x)$ plotted against x for increasing values of the scaled time $K_n t$ and for (a) $n = 4$, (b) $n = 3$, (c) $n = 2.5$.

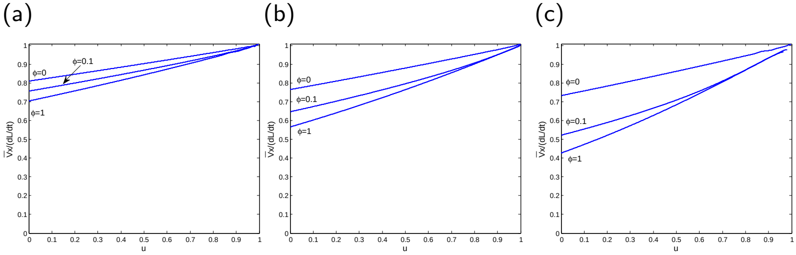
Variation of ϕ



Partially open fracture ($\beta = 0.5$) propagating with fluid injected at the fracture entry at a constant pressure for (i) $\phi = 0$, (ii) $\phi = 0.1$, (iii) $\phi = 0.5$, (iv) $\phi = 1$ and for $n = 3$. (a) The half-width of the fracture plotted against x for the time scale $K_n t = 20$ (b) The length of the fracture plotted against $K_n t$.

Width averaged fluid velocity

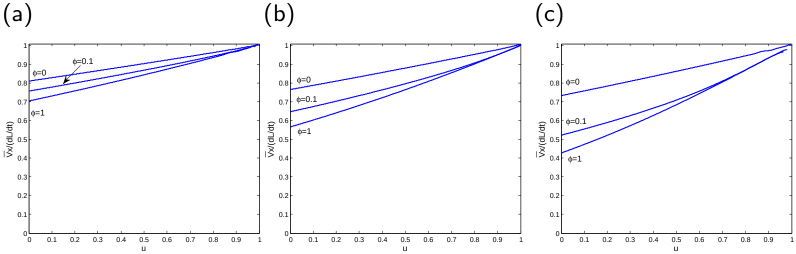
$$\textbf{Velocity ratio}: \frac{\overline{v}_x}{dL/dt} = -f^{n-1}\left(1+\phi\left(\frac{f(0)}{\beta f}\right)^2\right)\frac{df}{du}$$



Velocity ratio curves $\frac{\overline{v}_x}{dL/dt}$ plotted againt u for a partially open fracture ($\beta = 0.5$) propagating with fluid injected at the fracture entry at a constant pressure and for (a) $n = 4$, (b) $n = 3$, (c) $n = 2.5$.

Width averaged fluid velocity

$$\textbf{Velocity ratio}: \frac{\overline{v}_x}{dL/dt} = -f^{n-1}\left(1+\phi\left(\frac{f(0)}{\beta f}\right)^2\right)\frac{df}{du} = (1-A)u + A$$



Velocity ratio curves $\frac{\overline{v}_x}{dL/dt}$ plotted againt u for a partially open fracture ($\beta = 0.5$) propagating with fluid injected at the fracture entry at a constant pressure and for (a) $n = 4$, (b) $n = 3$, (c) $n = 2.5$.

Approximate analytical solution

- Problem for $2 < n < 5$: $f^n + pf^{n-2} + q = 0$,

  where $p = \dfrac{n\phi f^2(0)}{\beta^2(n-2)}$, $q = n\Big(\dfrac{(1-A)}{2}u^2 + Au - \dfrac{(A+1)}{2}\Big)$

  and $f(0) = \Big(\dfrac{(A+1)}{2}\Big)^{\frac{1}{n}}\left[\dfrac{n(n-2)\beta^2}{\beta^2(n-2)+\phi n}\right]^{\frac{1}{n}}$.

- Special case of n=3: $f^3 + pf + q = 0$,

$$f(u) = \frac{(\sqrt{3}\sqrt{4p^3+27q^2} - 9q)^{1/3}}{2^{1/3}3^{2/3}} - \frac{\left(\frac{2}{3}\right)^{1/3} p}{(\sqrt{3}\sqrt{4p^3+27q^2} - 9q)^{1/3}}.$$

- Special case of n=4: $f^4 + pf^2 + q = 0$,

$$f(u) = \frac{\sqrt{-p+\sqrt{p^2-4q}}}{\sqrt{2}}$$

- General case $2 < n < 5$:

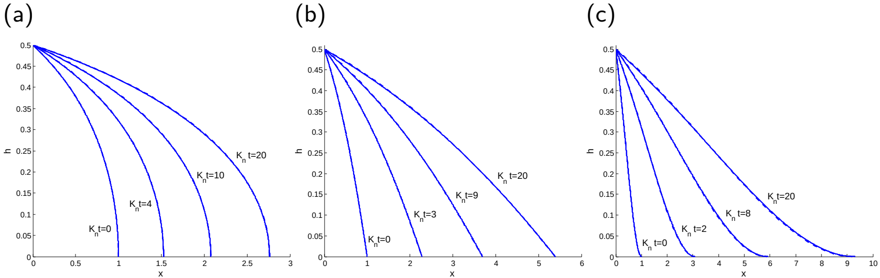
$$f(u) = \left[\frac{[(A+1) - 2Au - (1-A)u^2]\beta^2 n(n-2) f^2(u)}{2[\beta^2(n-2)f^2(u) + \phi n f^2(0)]} \right]^{\frac{1}{n}}$$

$$f_{i+1} = \left[\frac{[(A+1) - 2Au - (1-A)u^2]\beta^2 n(n-2) f_i^2}{2[\beta^2(n-2)f_i^2 + \phi n f^2(0)]} \right]^{\frac{1}{n}}, \quad i = 0, 1, 2, \ldots s,$$

$$A_{i+1} = \left[2^{\frac{1}{n}} \beta^2 n(n-2)[\beta^2(n-2) + \phi n]^{\frac{2}{n}} \right]^{\frac{1}{n}} \int_0^1 (1-u)^{\frac{1}{n}} \times$$

$$\left(\frac{f_i^2}{2^{\frac{2}{n}} \beta^2 (n-2)[\beta^2(n-2) + \phi n]^{\frac{2}{n}} f_i^2 + \phi n (A_i + 1)^{\frac{2}{n}} [n(n-2)\beta^2]^{\frac{2}{n}}} \right)^{\frac{1}{n}} du,$$

$s + 1 =$ No. of iterations for convergence to be achieved



Comparison of the numerical (—) and the approximate analytical (– – –) half-width solutions plotted against x for increasing time scales $K_n t$ and for (a) $n = 4$ (b) $n = 3$ (c) $n = 2.5$.

Conclusions

- Earlier paper: the linear crack model was found to give both fluid injection and extraction solutions.
- The hyperbolic crack law model was found to admit only one solution of fluid injected at constant pressure at the fracture entry.
- All solutions for the linear crack law model were found converge to the constant pressure solution.
- An analytical solution could not be derived. A numerical solution was therefore investigated and obtained.
- The width averaged fluid velocity was obtained to increase approx linearly along the fracture length.
- The approximate analytical solution may be required in practice.

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Thank You