An Operator Splitting Based Stochastic Galerkin Method for Nonlinear Systems of Hyperbolic Conservation Laws with Uncertainty

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Supported by NSF

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Conservation/Balance Laws with Uncertainties

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U}, x, \boldsymbol{z})_x = \boldsymbol{R}(\boldsymbol{U}, x, \boldsymbol{z}), \quad x \in \mathbb{R},\ t > 0,\ \boldsymbol{z} \in \Omega \subset \mathbb{R}^d$$

$\boldsymbol{U} = \boldsymbol{U}(x, t, \boldsymbol{z})$ is the unknown vector function

x: spatial variable

t: time variable

$\boldsymbol{z}$: random variable

$\boldsymbol{F}$: flux vector function

$\boldsymbol{R}$: source term

Uncertainties can appear in the source terms, equations of state, initial or boundary data due to empirical approximations or measuring errors

Quantifying Uncertainties – gPC Approach

Polynomial chaos or generalized polynomial chaos (gPC) approach:

• **Non-intrusive gPC method** – solves the original problem at selected sampling points, thus one can use the deterministic code, and then use interpolation and quadrature rules to numerically evaluate the statistical moments

[Xiu, Hesthaven; 2005]

[Mishra, Schwab, Sukys; 2012]

• **Intrusive gPC method** – uses the Galerkin approximation, which results in a system of deterministic equations, solving which will give the stochastic moments of the solution of the original uncertain problem

– Pros: lower computational cost; theoretical advantages;

[Elman, Miller, Phipps, Tuminaro; 2011]

– Cons: extra efforts are needed in order to obtain well-behaved discrete systems

[Xui; 2010]

[Tryoen, Le Maitre, Ndjinga, Ern; 2010]

[Després, Poëtte, Lucor; 2013]

[Pettersson, Iaccarino, Nordström; 2014, 2015]

[Hu, Jin, Xiu; 2015]

The gPC-SG Method – An Overview

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U}, x, \boldsymbol{z})_x = \boldsymbol{R}(\boldsymbol{U}, x, \boldsymbol{z}), \quad x \in \mathbb{R},\ t > 0,\ \boldsymbol{z} \in \Omega \subset \mathbb{R}^d$$

The solution is sought in terms of an orthogonal polynomial series in $\boldsymbol{z}$

$$\boldsymbol{U}(\boldsymbol{x}, t, \boldsymbol{z}) \approx \boldsymbol{U}_N(\boldsymbol{x}, t, \boldsymbol{z}) = \sum_{i=0}^{M-1} \hat{\boldsymbol{U}}_i(\boldsymbol{x}, t)\Phi_i(\boldsymbol{z}), \quad M = \binom{d+N}{d}$$

• $\{\Phi_i(\boldsymbol{z})\}$ are multidimensional polynomials of degree up to N of $\boldsymbol{z}$:

$$\int_\Omega \Phi_i(\boldsymbol{z})\Phi_\ell(\boldsymbol{z})\mu(\boldsymbol{z})\, d\boldsymbol{z} = \delta_{i\ell}, \qquad 0 \le i, \ell \le M-1 \ M = \dim\left(\mathbb{P}_N^d\right)$$

• $\mu(\boldsymbol{z})$: probability density function of $\boldsymbol{z}$

• $\delta_{i\ell}$: Kronecker symbol

• The choice of the orthogonal polynomials depends on the distribution function of $\boldsymbol{z}$. For example:

– a Gaussian distribution defines the Hermite polynomials

– a uniform distribution defines the Legendre polynomials

The gPC-SG method seeks to satisfy the system in a weak form by ensuring that the residual is orthogonal to the gPC polynomial space.

Substituting
$$\boldsymbol{U}_N(\boldsymbol{x},t,\boldsymbol{z}) = \sum_{i=0}^{M-1} \widehat{\boldsymbol{U}}_i(\boldsymbol{x},t)\Phi_i(\boldsymbol{z})$$

into the governing system
$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U},x,\boldsymbol{z})_x = \boldsymbol{R}(\boldsymbol{U},x,\boldsymbol{z})$$

and using the Galerkin projection yield
$$(\widehat{\boldsymbol{U}}_i)_t + (\widehat{\boldsymbol{F}}_i)_x = \widehat{\boldsymbol{R}}_i, \quad 0 \le i \le M-1$$

where
$$\widehat{\boldsymbol{F}}_i = \int_{\Omega} \boldsymbol{F}\left(\sum_{j=0}^{M-1} \widehat{\boldsymbol{U}}_j(\boldsymbol{x},t)\Phi_j(\boldsymbol{z}), x, \boldsymbol{z}\right)\Phi_i(\boldsymbol{z})\mu(\boldsymbol{z})\,d\boldsymbol{z}$$
$$\widehat{\boldsymbol{R}}_i = \int_{\Omega} \boldsymbol{R}\left(\sum_{j=0}^{M-1} \widehat{\boldsymbol{U}}_i(\boldsymbol{x},t)\Phi_i(\boldsymbol{z}), x, \boldsymbol{z}\right)\Phi_i(\boldsymbol{z})\mu(\boldsymbol{z})\,d\boldsymbol{z}$$

The gPC-SG Method – Challenges

$\boldsymbol{U}_t+\boldsymbol{F}(\boldsymbol{U},x,\boldsymbol{z})_x=\boldsymbol{R}(\boldsymbol{U},x,\boldsymbol{z})$	$(\widehat{\boldsymbol{U}}_i)_t+(\widehat{\boldsymbol{F}}_i)_x=\widehat{\boldsymbol{R}}_i \quad 0\le i\le M-1$
Linear Hyperbolic	Hyperbolic
Nonlinear Symmetric	Hyperbolic
Nonlinear Nonsymmetric	?

• Our goal: *Introduce an operator splitting for the original hyperbolic system, which will guarantee that the gPC-SG discretization of each of the split subsystems always results in a globally hyperbolic system*

• Our strategy: generic, but the splitting is problem specific

• Our examples: the compressible Euler equations and the shallow water equations

1-D Compressible Euler Equations

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + (\rho u^2 + p)_x = 0 \\ E_t + (u(E+p))_x = 0 \end{cases}$$

- ρ: density
- u: velocity, $m = \rho u$: momentum
- E: total energy
- p: pressure with the equation of state $p = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right)$
- γ: specific heat ratio

We assume here that the data may depend on random variable $\boldsymbol{z}$, i.e.,

$$\rho(x,0,\boldsymbol{z}) = \rho_0(x,\boldsymbol{z}),\ u(x,0,\boldsymbol{z}) = u_0(x,\boldsymbol{z}),\ p(x,0,\boldsymbol{z}) = p_0(x,\boldsymbol{z}),\ \gamma = \gamma(z)$$

Uncertainty may also arise from boundary data and other terms

1-D Euler Equations – Numerical Challenges

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + (\rho u^2 + p)_x = 0 \\ E_t + (u(E+p))_x = 0 \end{cases} \qquad \lambda = u, u \pm c, \quad c = \sqrt{\gamma p/\rho}$$

A direct application of the gPC-SG method to the system may fail due to the loss of hyperbolicity after the gPC-SG discretization

Operator Splitting:

• Linear hyperbolic system

• Two degenerate nonlinear hyperbolic systems which are effectively scalar equations

The gPC-SG approximation is guaranteed to maintain the hyperbolicity for each of the subsystems

1-D Euler Equations – Operator Splitting

$$(I) \qquad \begin{cases} \rho_t + m_x = 0 \\ m_t + ((\gamma - 1)E + am)_x = 0 \\ E_t - (aE)_x = 0 \end{cases}$$

$$(II) \qquad \begin{cases} \rho_t = 0 \\ m_t + \left(\frac{3-\gamma}{2} \cdot \frac{m^2}{\rho} - am\right)_x = 0 \\ E_t = 0 \end{cases}$$

$$(III) \qquad \begin{cases} \rho_t = 0 \\ m_t = 0 \\ E_t + \left(\frac{m}{\rho}\left[\gamma E - \frac{\gamma - 1}{2} \cdot \frac{m^2}{\rho}\right] + aE\right)_x = 0 \end{cases}$$

- We choose:

  $-|a| \le u - c < u + c \le |a|$: subcharacteristic condition

  $a = \pm \sup(\max\{|u| + c, \gamma u, (3-\gamma)u\})$: convection coefficient should not change sign

Strang Splitting

$$\boldsymbol{U}_t + \boldsymbol{F}_I(\boldsymbol{U})_x = \boldsymbol{0} \qquad \rightarrow \qquad \mathcal{S}_I$$

$$\boldsymbol{U}_t + \boldsymbol{F}_{II}(\boldsymbol{U})_x = \boldsymbol{0} \qquad \rightarrow \qquad \mathcal{S}_{II}$$

$$\boldsymbol{U}_t + \boldsymbol{F}_{III}(\boldsymbol{U})_x = \boldsymbol{0} \qquad \rightarrow \qquad \mathcal{S}_{III}$$

Here

$$\boldsymbol{U} = \begin{pmatrix} r \\ m \\ E \end{pmatrix}, \quad \boldsymbol{F}_I = \begin{pmatrix} m \\ (\gamma - 1)E + am \\ -aE \end{pmatrix}, \quad \boldsymbol{F}_{II} = \begin{pmatrix} 0 \\ \frac{3-\gamma}{2} \cdot \frac{m^2}{\rho} - am \\ 0 \end{pmatrix}$$

$$\boldsymbol{F}_{III} = \begin{pmatrix} 0 \\ 0 \\ \frac{m}{\rho}\left[\gamma E - \frac{\gamma-1}{2} \cdot \frac{m^2}{\rho}\right] + aE \end{pmatrix}$$

- Assume that the solution of the original system is available at time t
- Introduce a (small) time step Δt
- One time step of the second-order Strang splitting method:

$$\boldsymbol{U}(\boldsymbol{x}, t + \Delta t, \boldsymbol{z}) = \mathcal{S}_I(\Delta t/2)\mathcal{S}_{II}(\Delta t/2)\mathcal{S}_{III}(\Delta t)\mathcal{S}_{II}(\Delta t/2)\mathcal{S}_I(\Delta t/2)\boldsymbol{U}(\boldsymbol{x}, t, \boldsymbol{z})$$

Operator Splitting – Numerical Validation

- We consider the Sod shock tube problem – pure deterministic problem:

$$\rho_0(x) = \begin{cases} 1, & x < 0.5, \\ 0.125, & x > 0.5, \end{cases} \quad u_0(x) \equiv 0, \quad p_0(x) = \begin{cases} 1, & x < 0.5 \\ 0.1, & x > 0.5 \end{cases}$$

- We run numerical simulations for both the unsplit and split systems

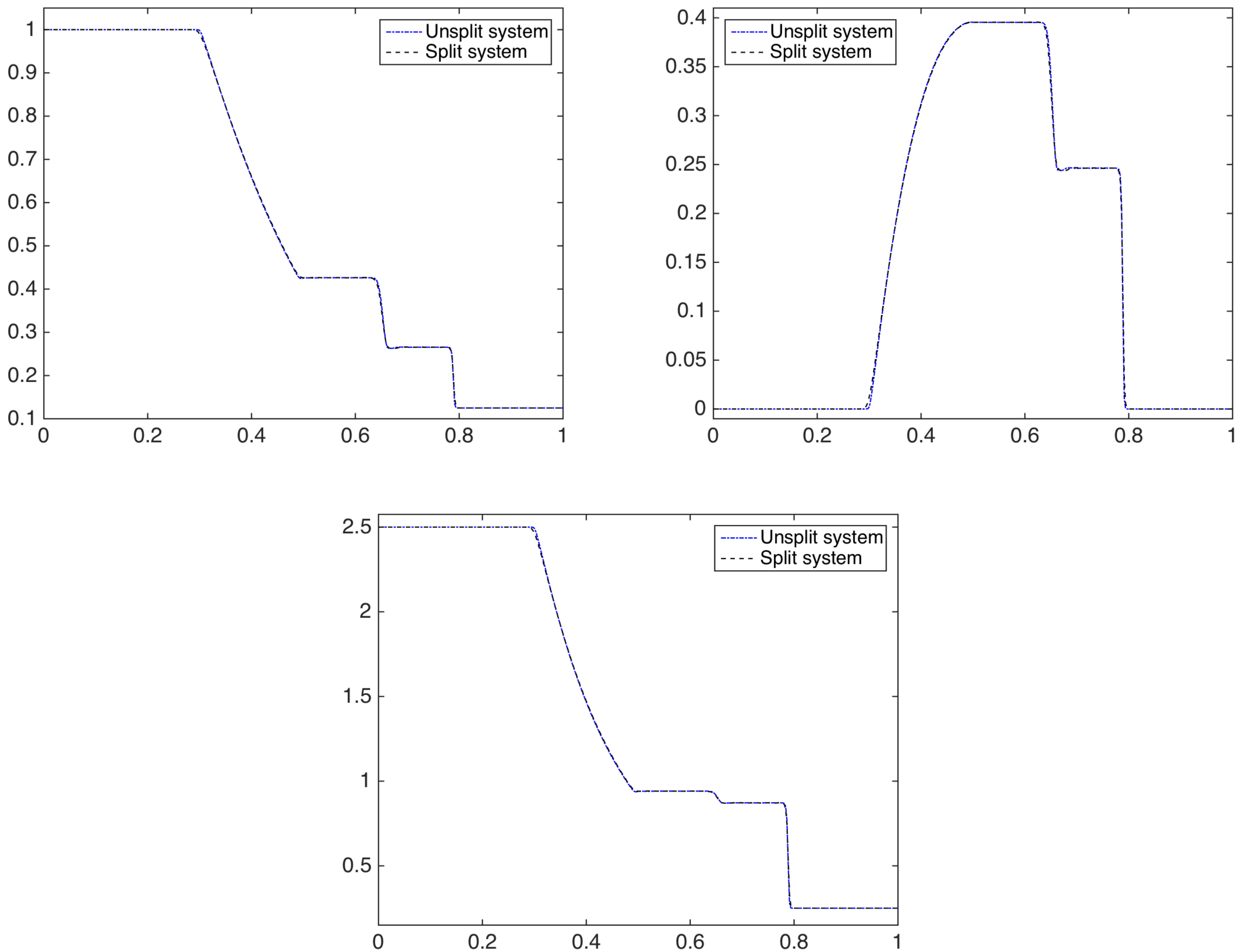
- We compare the results computed by the central-upwind scheme

– computational domain [0,1]

– non-reflecting boundary conditions

– uniform grid with $\Delta x = 1/400$

– final time $t = 0.1644$



ρ (top left), m (top right) and E (bottom)

1-D Euler Equations – The gPC-SG Approximation

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + ((\gamma - 1)E + am)_x = 0 \\ E_t - (aE)_x = 0 \end{cases} \qquad \begin{cases} \rho_t = 0 \\ m_t + \left(\dfrac{3-\gamma}{2} \cdot \dfrac{m^2}{\rho} - am \right)_x = 0 \\ E_t = 0, \end{cases}$$

$$\begin{cases} \rho_t = 0 \\ m_t = 0 \\ E_t + \left(\dfrac{m}{\rho} \left[\gamma E - \dfrac{\gamma - 1}{2} \cdot \dfrac{m^2}{\rho} \right] + aE \right)_x = 0 \end{cases}$$

We define the gPC expansions of ρ, m, E and γ in the following form:

$$\rho_N(x,t,\boldsymbol{z}) = \sum_{i=0}^{N} \widehat{\rho}_i(x,t)\Phi_i(\boldsymbol{z}), \quad m_N(x,t,\boldsymbol{z}) = \sum_{i=0}^{N} \widehat{m}_i(x,t)\Phi_i(\boldsymbol{z}),$$
$$E_N(x,t,\boldsymbol{z}) = \sum_{i=0}^{N} \widehat{E}_i(x,t)\Phi_i(\boldsymbol{z}), \quad \gamma_N(\boldsymbol{z}) = \sum_{i=0}^{N} \widehat{\gamma}_i\Phi_i(\boldsymbol{z})$$

substitute them into the systems and derive the gPC-SG approximation

...

We define ...

$$\gamma_N(\boldsymbol{z}) - 1 = \sum_{i=0}^{N} \hat{\hat{\gamma}}_i \Phi_i(\boldsymbol{z}), \quad \frac{3-\gamma_N(\boldsymbol{z})}{2} = \sum_{i=0}^{N} \hat{\hat{\hat{\gamma}}}_i \Phi_i(\boldsymbol{z}),$$

$$\left(\frac{m^2}{\rho}\right)_N (x,t,\boldsymbol{z}) = \sum_{i=1}^{N} \hat{\psi}_i(x,t)\Phi_i(\boldsymbol{z}),$$

$$\left(\frac{\gamma m}{\rho}\right)_N (x,t,\boldsymbol{z}) = \sum_{i=1}^{N} \hat{\hat{\psi}}_i(x,t)\Phi_i(\boldsymbol{z}),$$

$$\left(\frac{(\gamma-1)m}{\rho}\right)_N (x,t,\boldsymbol{z}) = \sum_{i=1}^{N} \hat{\hat{\hat{\psi}}}_i(x,t)\Phi_i(\boldsymbol{z}).$$

For example, $\hat{\psi}_i$ can be computed by using $\rho\psi = m^2$, namely,

$$\sum_{k,\ell=0}^{N} \hat{\psi}_k \hat{\rho}_\ell S_{ik\ell} = \sum_{k,\ell=0}^{N} \hat{m}_k \hat{m}_\ell S_{ik\ell}, \quad i = 0,\ldots,N$$

$$S_{ik\ell} = \int_\Omega \Phi_i(\boldsymbol{z})\Phi_k(\boldsymbol{z})\Phi_\ell(\boldsymbol{z})\mu(\boldsymbol{z})\,d\boldsymbol{z}$$ is computed once

... after implementing the Galerkin projection we obtain the corresponding three systems for the gPC coefficients $i = 0, \ldots, N$:

$$
(I) \quad \begin{cases} (\widehat{\rho}_i)_t + (\widehat{m}_i)_x = 0 \\ (\widehat{m}_i)_t + \displaystyle\sum_{k,\ell=0}^{N} \hat{\hat{\gamma}}_k (\widehat{E}_\ell)_x S_{k\ell i} + (a\widehat{m}_i)_x = 0 \\ (\widehat{E}_i)_t - (a\widehat{E}_i)_x = 0 \end{cases}
$$

$$
(II) \quad \begin{cases} (\widehat{\rho}_i)_t = 0 \\ (\widehat{m}_i)_t + \displaystyle\sum_{k,\ell=0}^{N} \hat{\hat{\hat{\gamma}}}_k (\widehat{\psi}_\ell)_x S_{k\ell i} - (a\widehat{m}_i)_x = 0 \\ (\widehat{E}_i)_t = 0 \end{cases}
$$

$$
(III) \quad \begin{cases} (\widehat{\rho}_i)_t = 0 \\ (\widehat{m}_i)_t = 0 \\ (\widehat{E}_i)_t + \displaystyle\sum_{k,\ell=0}^{N} (\hat{\hat{\psi}}_k \widehat{E}_\ell)_x S_{k\ell i} - \displaystyle\sum_{k,\ell=0}^{N} (\hat{\hat{\hat{\psi}}}_k \widehat{\psi}_\ell)_x S_{k\ell i} + (a\widehat{E}_i)_x = 0 \end{cases}
$$

Strang Splitting + Spatial Discretization

For each $i = 0, \ldots, N$:

$(\widehat{\boldsymbol{U}}_i)_t + (\widehat{\boldsymbol{F}}_I)(\widehat{\boldsymbol{U}}_i)_x = \mathbf{0} \quad \rightarrow \quad \mathcal{S}_I$ solution operator (CU scheme)

$(\widehat{\boldsymbol{U}}_i)_t + \widehat{\boldsymbol{F}}_{II}(\widehat{\boldsymbol{U}}_i)_x = \mathbf{0} \quad \rightarrow \quad \mathcal{S}_{II}$ solution operator (CU scheme)

$(\widehat{\boldsymbol{U}}_i)_t + \widehat{\boldsymbol{F}}_{III}(\widehat{\boldsymbol{U}}_i)_x = \mathbf{0} \quad \rightarrow \quad \mathcal{S}_{III}$ solution operator (CU scheme)

- Assume that the solution of the original system is available at time t
- Introduce a (small) time step Δt
- One time step of the second-order Strang splitting method:

$$\boldsymbol{U}_i(\boldsymbol{x}, t+\Delta t, \boldsymbol{z}) = \mathcal{S}_I(\Delta t/2)\mathcal{S}_{II}(\Delta t/2)\mathcal{S}_{III}(\Delta t)\mathcal{S}_{II}(\Delta t/2)\mathcal{S}_I(\Delta t/2)\boldsymbol{U}_i(\boldsymbol{x}, t, \boldsymbol{z})$$

Central-Upwind Schemes

- Godunov-type finite-volume methods

- Central: Riemann-problem-solver-free methods designed without tracking complicated nonlinear waves

- Upwind: Use some information on wave propagation to reduce numerical dissipation and thus enhance the resolution of nonsmooth waves

- Can be applied as a "black-box" solver to (multidimensional) hyperbolic systems of PDEs

- Robust, efficient and highly accurate

[Kurganov, Tadmor; 2000]

[Kurganov, Petrova; 2001]

[Kurganov, Noelle, Petrova; 2001]

[Kurganov, Tadmor; 2002]

[Kurganov, Petrova; 2005]

[Kurganov, Lin; 2007]

[Kurganov, Prugger, Wu; preprint]

Numerical Examples

• Three examples for the Sod problem
– Example 1 - Perturbed the initial conditions
– Example 2 - Perturbed γ
– Example 3 - Perturbed interface

• We always assume a 1-D random variable z obeying the uniform distribution on $[-1, 1]$, thus the Legendre polynomials are used as the gPC basis

• The mean and standard deviation of the computed solution $\boldsymbol{U}$, which are shown in the Figures below, are given by

$$\mathbb{E}[\boldsymbol{U}] = \widehat{\boldsymbol{U}}_0, \quad \sigma[\boldsymbol{U}] = \sum_{i=1}^{N} (\widehat{\boldsymbol{U}}_i)^2,$$

where $\widehat{\boldsymbol{U}}_i, \; i = 0, \ldots, N$ are the computed gPC coefficients of $\boldsymbol{U}$.

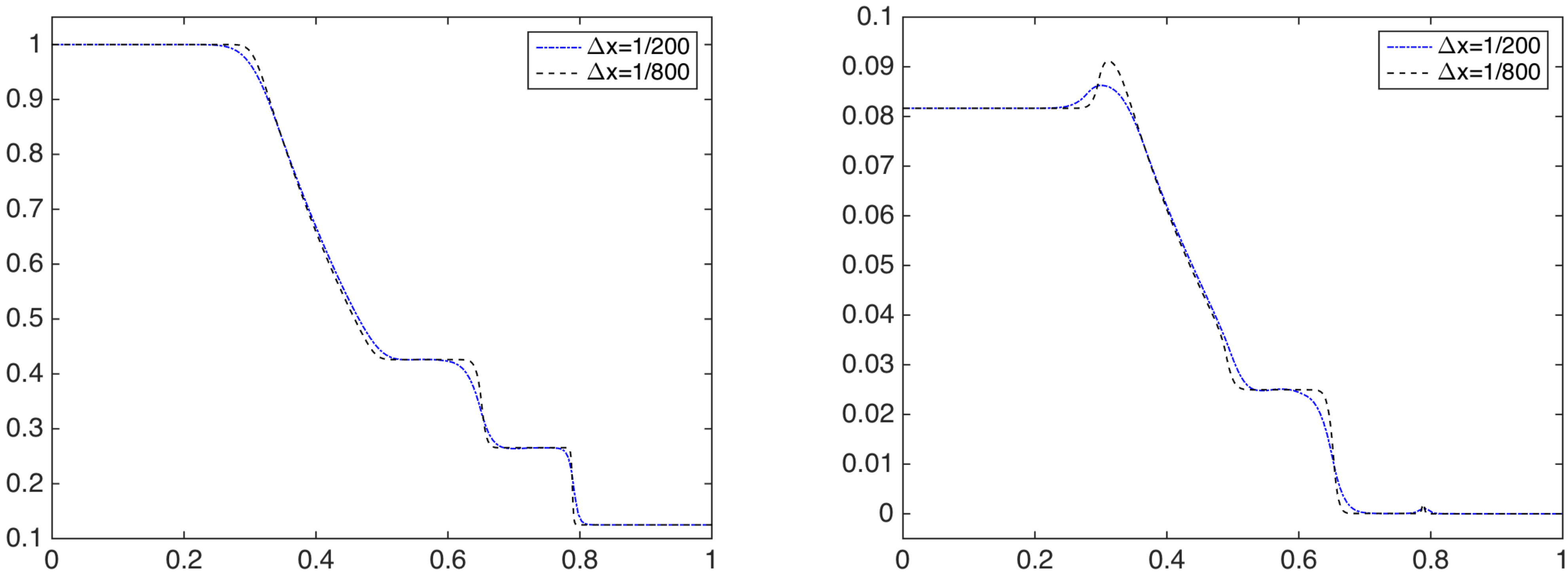
• In all the examples: Strang splitting + second-order semi-discrete central-upwind scheme was implemented for the spatial discretization

Example 1 – Perturbed Initial Data

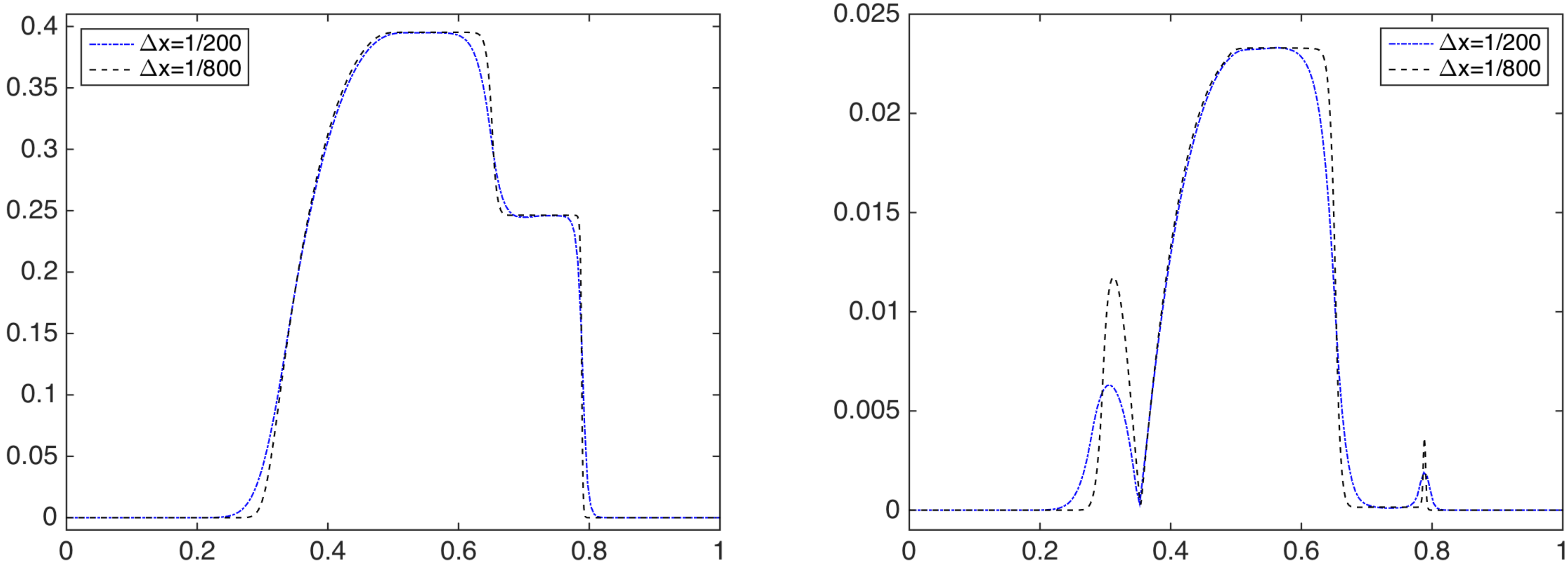
We consider the Sod shock tube problem with $\gamma = 1.4$ and subject to the following initial condition:

$$\rho_0(x,z) = \begin{cases} 1 + 0.1z, & x < 0.5, \\ 0.125, & x > 0.5, \end{cases} \quad u_0(x) \equiv 0, \quad p_0(x) = \begin{cases} 1, & x < 0.5 \\ 0.1, & x > 0.5 \end{cases}$$

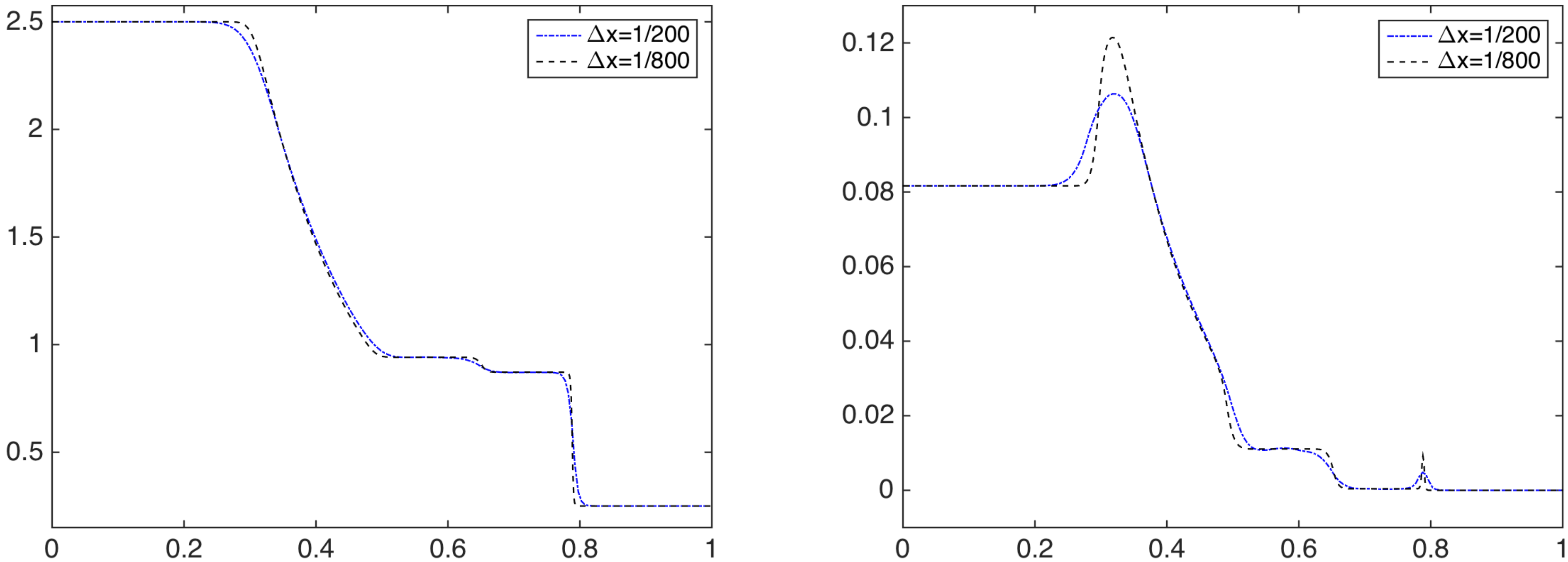
- Computational domain $[0,1]$
- Non-reflecting boundary conditions
- $N = 8$ – highest degree of the Legendre polynomials
- $\Delta x = 1/200$ and $\Delta x = 1/800$
- final time $t = 0.1644$



Mean (left) and standard deviation (right) of ρ



Mean (left) and standard deviation (right) of m



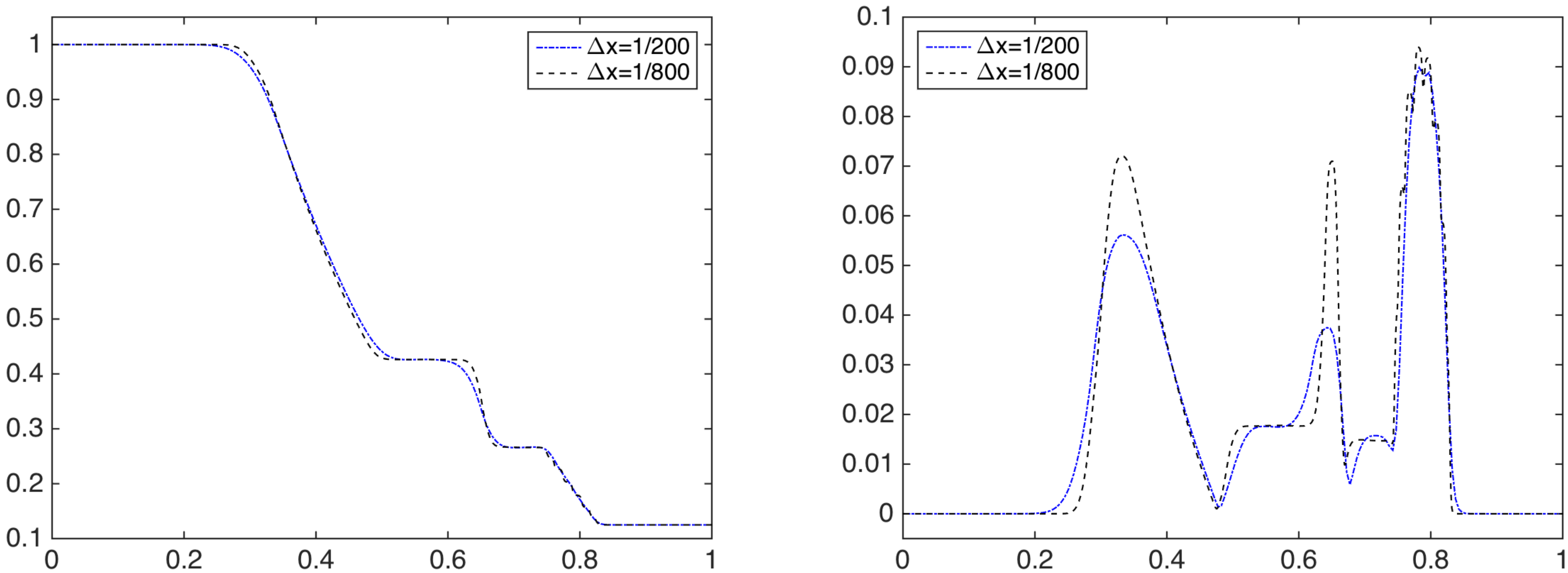
Mean (left) and standard deviation (right) of E

Example 2 – Perturbed γ

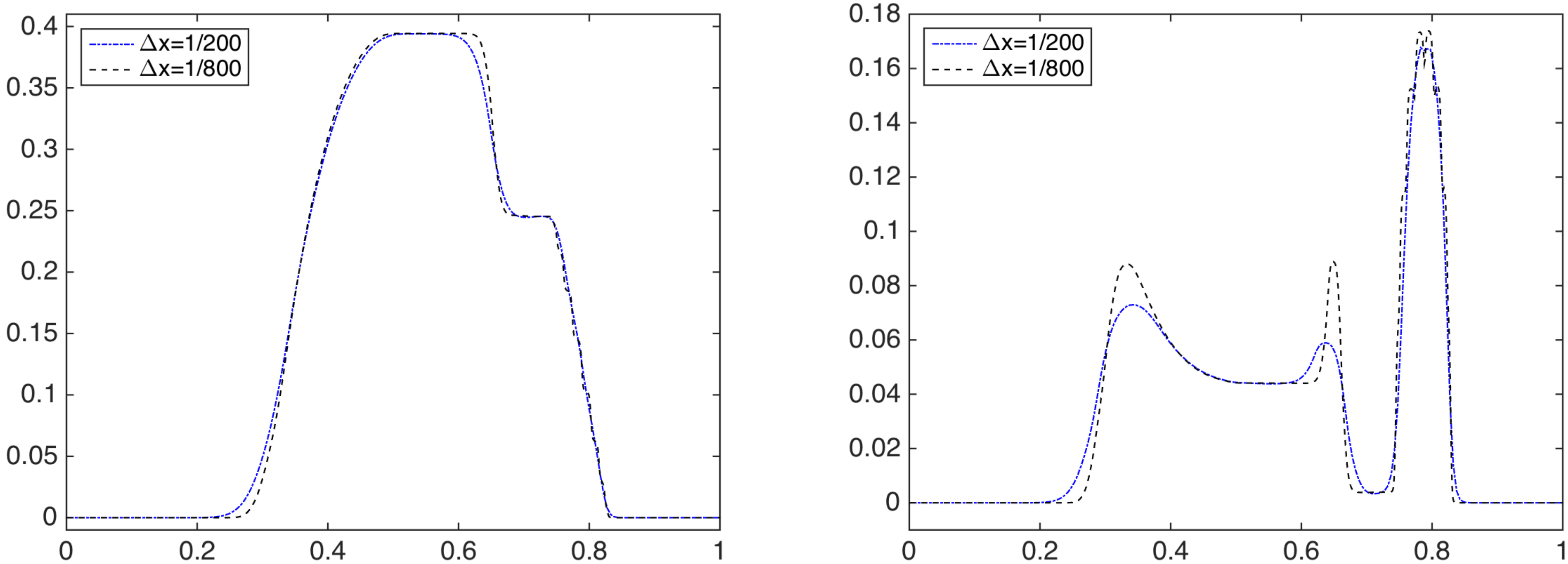
We consider the Sod shock tube problem with $\gamma(z) = 1.4 + 0.1z$ and subject to the following initial condition:

$$\rho_0(x,z) = \begin{cases} 1, & x < 0.5, \\ 0.125, & x > 0.5, \end{cases} \quad u_0(x) \equiv 0, \quad p_0(x) = \begin{cases} 1, & x < 0.5 \\ 0.1, & x > 0.5 \end{cases}$$

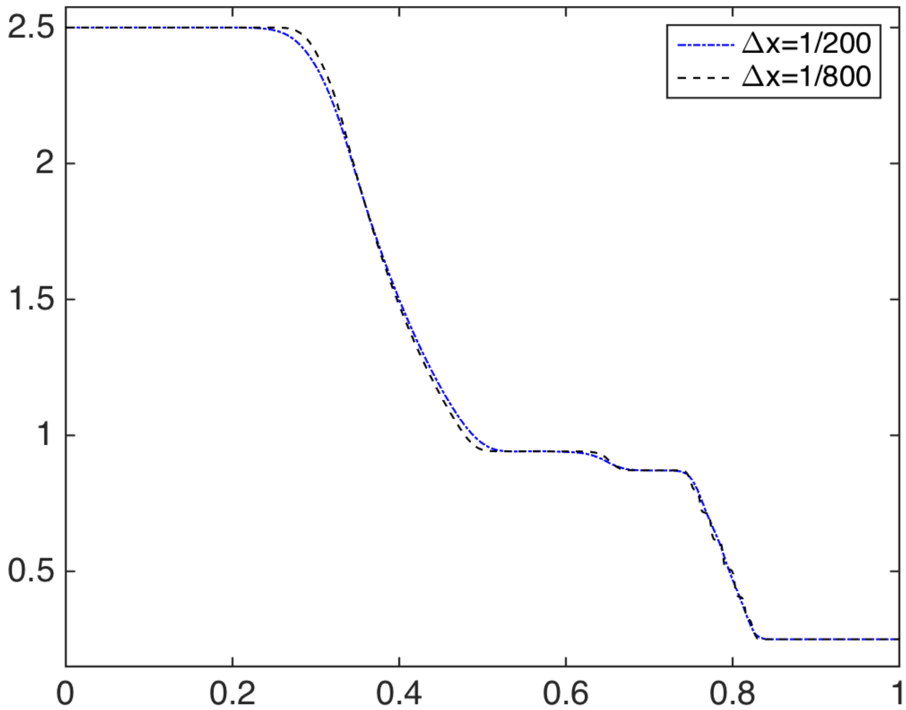
- Computational domain $[0, 1]$
- Non-reflecting boundary conditions
- $N = 8$ – highest degree of the Legendre polynomials
- $\Delta x = 1/200$ and $\Delta x = 1/800$
- final time $t = 0.1644$

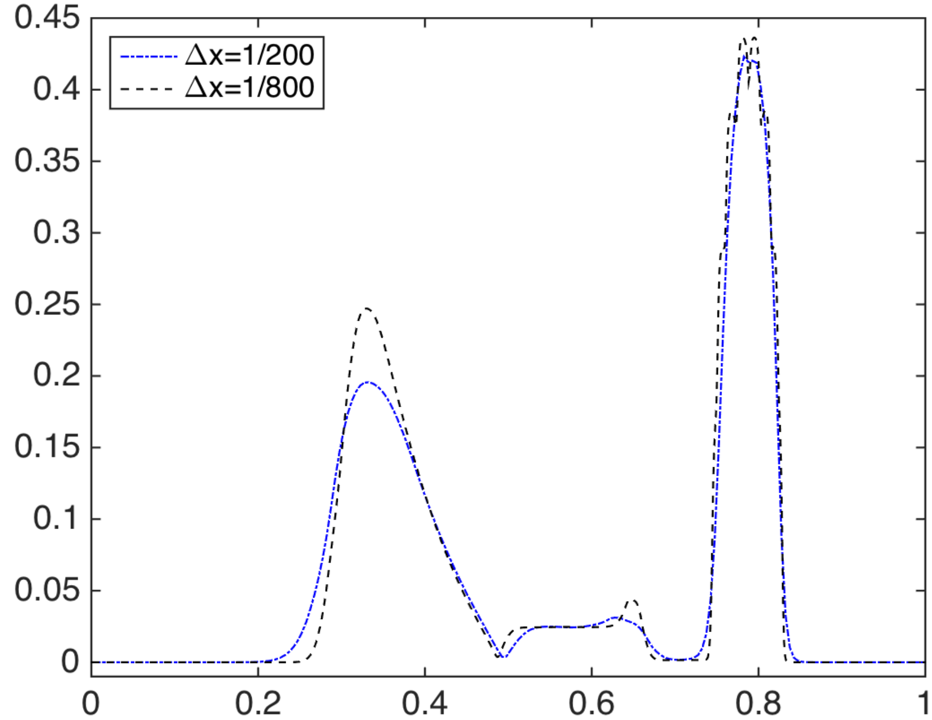


Mean (left) and standard deviation (right) of ρ



Mean (left) and standard deviation (right) of m





Mean (left) and standard deviation (right) of E

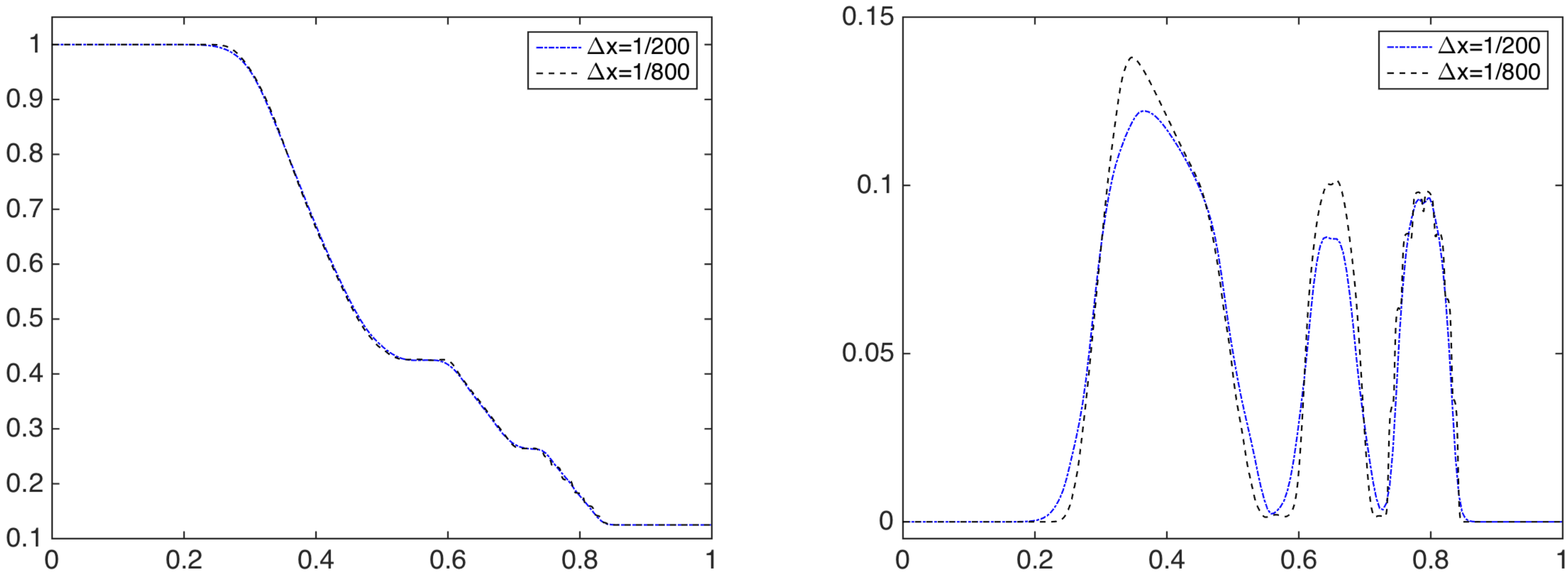
Example 3 – Perturbed Interface

We consider the Sod shock tube problem with $\gamma = 1.4$ and subject to the following initial condition:

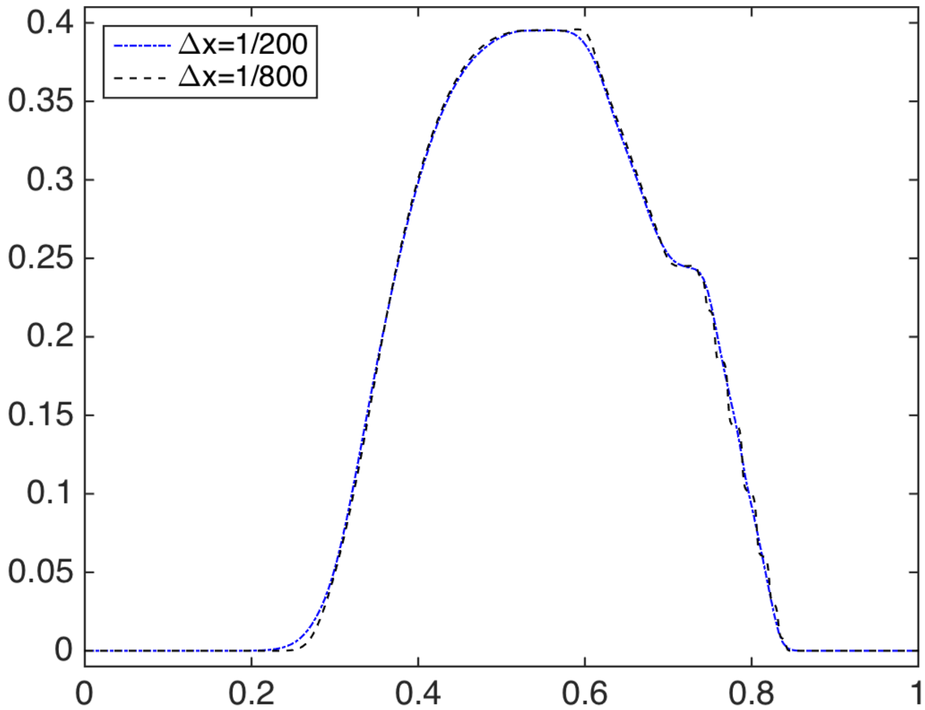
$$\rho_0(x,z) = \begin{cases} 1, & x < 0.5 + 0.05z \\ 0.125, & x > 0.5 + 0.05z \end{cases} \qquad u_0(x) \equiv 0$$

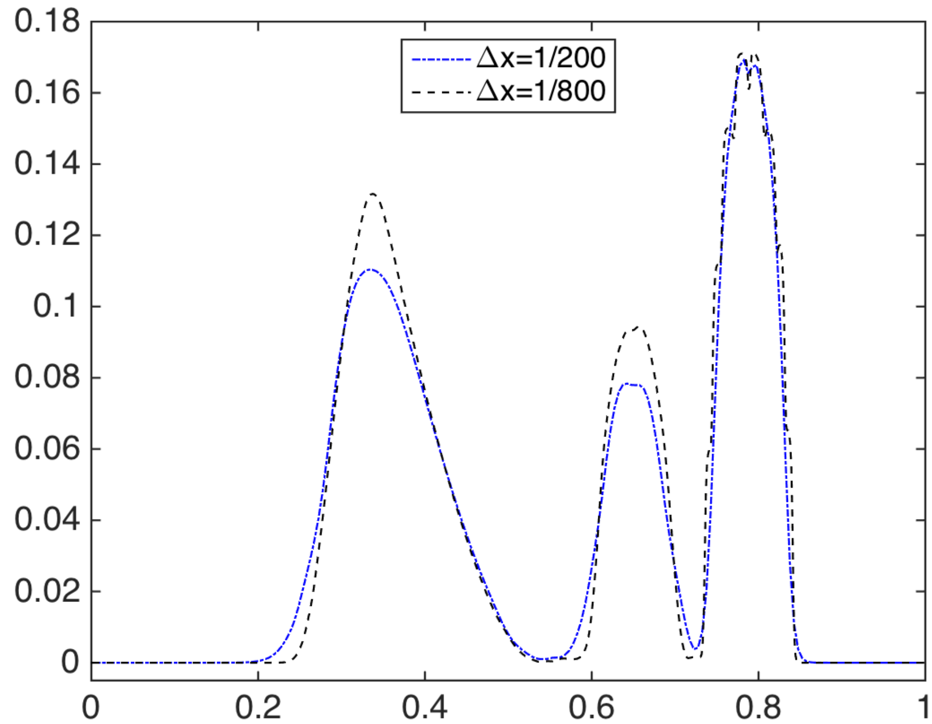
$$p_0(x) = \begin{cases} 1, & x < 0.5 + 0.05z \\ 0.1, & x > 0.5 + 0.05z \end{cases}$$

- Computational domain $[0,1]$
- Non-reflecting boundary conditions
- $N = 8$ – highest degree of the Legendre polynomials
- $\Delta x = 1/200$ and $\Delta x = 1/800$
- final time $t = 0.1644$

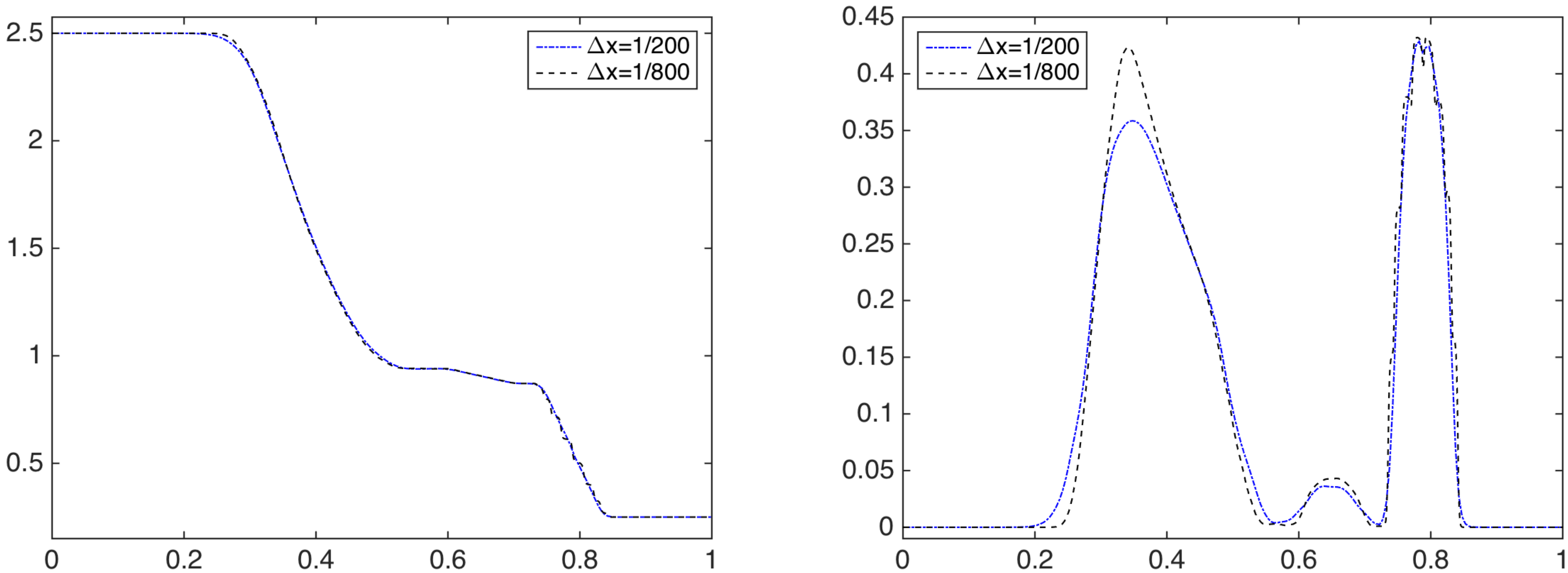


Mean (left) and standard deviation (right) of ρ





Mean (left) and standard deviation (right) of m



Mean (left) and standard deviation (right) of E