Multi-domain Bivariate Spectral Local Linearisation method for solving non-similar boundary layer partial differential equations

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Overview

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2. SOLUTION PROCEDURE
3. RESULTS AND DISCUSSION
4. CONCLUSION & FUTURE RESEARCH DIRECTION
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Aim & Objectives

Aim

To present an extension of the **Bivariate Spectral Local Linearisation Method (BSLLM)**[1] for solving non-similar nonlinear PDEs over **large time intervals**.

- BSLLM uses Chebyshev-Gauss-Lobatto points (see [3, 4])
- Drawback of BSLLM - accuracy deteriorates over **large time intervals**.
- New approach termed - **Multi-domain Bivariate Spectral Local Linearisation Method** (MD-BSLLM)

Objectives

- Solve non-linear non-similar boundary layer equations over a large time domain using the MD-BSLLM.
- Validate the results using a series solution approach.
Solution Method

The solution approach involves

- Domain decomposition
- Linearisation and decoupling
- Bivariate interpolation.
- Pseudo-spectral approximation
Numerical Experiment

\[
\frac{\partial^3 f}{\partial \eta^3} + \frac{1}{4}(n+3)f \frac{\partial^2 f}{\partial \eta^2} - \frac{1}{2}(n+1) \left( \frac{\partial f}{\partial \eta} \right)^2 + \zeta \frac{\partial^2 f}{\partial \eta^2} + (1-w)g + wh
\]

\[
= \frac{1}{4}(1-n)\zeta \left[ \frac{\partial f}{\partial \eta} \frac{\partial^2 f}{\partial \zeta \partial \eta} - \frac{\partial^2 f}{\partial \eta^2} \frac{\partial f}{\partial \zeta} \right], \tag{1}
\]

\[
\frac{1}{Pr} \frac{\partial^2 g}{\partial \eta^2} + \frac{1}{4}(n+3)f \frac{\partial g}{\partial \eta} + \zeta \frac{\partial g}{\partial \eta} = \frac{1}{4}(1-n)\zeta \left[ \frac{\partial f}{\partial \eta} \frac{\partial g}{\partial \zeta} - \frac{\partial g}{\partial \eta} \frac{\partial f}{\partial \zeta} \right], \tag{2}
\]

\[
\frac{1}{Sc} \frac{\partial^2 h}{\partial \eta^2} + \frac{1}{4}(n+3)f \frac{\partial h}{\partial \eta} + \zeta \frac{\partial h}{\partial \eta} = \frac{1}{4}(1-n)\zeta \left[ \frac{\partial f}{\partial \eta} \frac{\partial h}{\partial \zeta} - \frac{\partial h}{\partial \eta} \frac{\partial f}{\partial \zeta} \right], \tag{3}
\]

subject to

\[f(\zeta, 0) = 0, \quad \frac{\partial f}{\partial \eta}(\zeta, 0) = 0, \quad g(\zeta, 0) = h(\zeta, 0) = 1, \quad \frac{\partial f}{\partial \eta}(\zeta, \infty) = g(\zeta, \infty) = h(\zeta, \infty) = 0.\]

- Formulated and Solved by Hussain et.al. [2] using finite-difference based Keller-box technique
- Results were validated using Series Solutions, for small and large \(\zeta\)
The patching condition requires that

\[ f^{(l)}(\eta, \zeta_{l-1}) = f^{(l-1)}(\eta, \zeta_{l-1}), \quad \eta \in [a, b], \]

where \( f^{(l)}(\eta, \zeta) \) denotes the solution of equation (2) at each sub-interval \( \Gamma_l \).
Linearisation and Decoupling

- The PDEs are linearised using the **quasi-linearisation method**.
  - an iterative procedure based on the **Taylor-series expansion** about the previous estimates of the solution.
- Applying the **quasi-linearisation method** on one function at a time, results in three **decoupled linear PDEs**.
- In each sub-interval \([\zeta_{l-1}, \zeta_{l}]\), these decoupled linear PDEs can be solved iteratively in a sequential manner until the desired solution is obtained.

\[
\begin{align*}
\beta_0, r \frac{\partial^3 f_{r+1}^{(l)}}{\partial \eta^3} + \beta_1, r \frac{\partial^2 f_{r+1}^{(l)}}{\partial \eta^2} + \beta_2, r \frac{\partial f_{r+1}^{(l)}}{\partial \eta} + \beta_3, r f_{r+1}^{(l)} + \beta_4, r \frac{\partial f_{r+1}^{(l)}}{\partial \xi} + \beta_5, r \frac{\partial f_{r+1}^{(l)}}{\partial \xi} &= R_{f, r}, \\
\sigma_1, r \frac{\partial^2 g_{r+1}^{(l)}}{\partial \eta^2} + \sigma_2, r \frac{\partial g_{r+1}^{(l)}}{\partial \eta} + \sigma_3, r g_{r+1}^{(l)} + \sigma_4, r \frac{\partial g_{r+1}^{(l)}}{\partial \xi} &= R_{g, r}, \\
\omega_1, r \frac{\partial^2 h_{r+1}^{(l)}}{\partial \eta^2} + \omega_2, r \frac{\partial h_{r+1}^{(l)}}{\partial \eta} + \omega_3, r h_{r+1}^{(l)} + \omega_4, r \frac{\partial h_{r+1}^{(l)}}{\partial \xi} &= R_{h, r},
\end{align*}
\]
Pseudo-spectral Approximation

- Assume that the solution at each sub-interval $\Gamma_l$, denoted by $f^{(l)}(\eta, \zeta)$, can be approximated by a **bivariate Lagrange interpolation** polynomial of the form

$$f^{(l)}(\eta, \zeta) \approx \sum_{i=0}^{N_x} \sum_{j=0}^{N_t} f^{(l)}(\eta_i, \zeta_j) L_i(\eta) L_j(\zeta).$$  (8)

- Transpiration parameter derivative values are computed at the grid points $(\eta_i, \zeta_j)$:

$$\frac{\partial f^{(l)}}{\partial \zeta} \bigg|_{(\eta_i, \zeta_j)} = \left( \frac{2}{\zeta_l - \zeta_{l-1}} \right) \sum_{v=0}^{N_t} d_{jv} f^{(l)}(\eta_j, \zeta_v) = \left( \frac{2}{\zeta_l - \zeta_{l-1}} \right) \sum_{v=0}^{N_t} d_{jv} F^{(l)}_v.$$  (9)

- The $n$th order space derivative is defined as

$$\frac{\partial^n f^{(l)}}{\partial \eta^n} \bigg|_{(\eta_i, \zeta_j)} = \left( \frac{2}{\eta_\infty} \right)^n \sum_{\rho=0}^{N_x} D_{ip}^n f^{(l)}(\eta_\rho, \zeta_j) = \left( \frac{2}{\eta_\infty} \right)^n D^n F^{(l)}_j, \quad j = 0, 1, 2, \ldots, N_t,$$  (10)

where the vector $F^{(l)}_j$ is defined as

$$F^{(l)}_j = [f^{(l)}(\eta_0, \zeta_j), f^{(l)}(\eta_1, \zeta_j), \ldots, f^{(l)}(\eta_{N_x}, \zeta_j)]^T.$$  (11)
Pseudo-spectral Approximation

- Substituting equations (9) and (10) into equation (5), we get

\[
\left[ \beta_{0,r} D^3 + \beta_{1,r} D^2 + \beta_{2,r} D + \beta_{3,r} \right] F_{r+1,j}^{(l)} + \beta_{4,r} \sum_{v=0}^{N_t} d_{jv} F_{r+1,v}^{(l)} + \beta_{5,r} \sum_{v=0}^{N_t} d_{jv} D F_{r+1,v}^{(l)} = R_{f,r}^{(l)}, \tag{12}
\]

for \( j = 0, 1, 2, \ldots, N_t \).

- The patching condition requires that

\[
f_{r+1}^{(l)}(\eta_i, \zeta(l-1,j)) = f_{r+1}^{(l-1)}(\eta_i, \zeta(l-1,j)), \quad \eta \in [a, b], \tag{13}
\]

- The initial unsteady solution of equation (5) when \( \zeta = 0 \) corresponds to \( t = t_{N_t} = -1 \).
- Equation (12) is evaluated for \( j = 0, 1, \ldots, N_t - 1 \)

\[
\left[ \beta_{0,r} D^3 + \beta_{1,r} D^2 + \beta_{2,r} D + \beta_{3,r} \right] F_{r+1,j}^{(l)} + \beta_{4,r} \sum_{v=0}^{N_t-1} d_{jv} F_{r+1,v}^{(l)} + \beta_{5,r} \sum_{v=0}^{N_t-1} d_{jv} D F_{r+1,v}^{(l)} = R_{1,j}^{(l)}, \tag{14}
\]

and

\[
R_{1,j}^{(l)} = R_{f,r}^{(l)} - \beta_{4,r} d_{jN_t} F_{N_t}^{(l)} - \beta_{5,r} d_{jN_t} D F_{N_t}^{(l)}. \]
Imposing boundary conditions for \( j = 0, 1, \cdots, N_t - 1 \), equation (14) can be expressed as the following

\[
N_t(N_x + 1) \times N_t(N_x + 1)
\]

matrix system

\[
\begin{bmatrix}
A_{0,0} & A_{0,1} & \cdots & A_{0,N_t-1} \\
A_{1,0} & A_{1,1} & \cdots & A_{1,N_t-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{N_t-1,0} & A_{N_t-1,1} & \cdots & A_{N_t-1,N_t-1}
\end{bmatrix}
\begin{bmatrix}
\mathbf{F}^{(l)}_0 \\
\mathbf{F}^{(l)}_1 \\
\vdots \\
\mathbf{F}^{(l)}_{N_t-1}
\end{bmatrix}
= \begin{bmatrix}
\mathcal{R}^{(l)}_{1,0} \\
\mathcal{R}^{(l)}_{1,1} \\
\vdots \\
\mathcal{R}^{(l)}_{1,N_t-1}
\end{bmatrix},
\tag{15}
\]

where

\[
A_{i,i} = \beta_0,r D^3 + \beta_1,r D^2 + \beta_2,r D + \beta_3,r I + \beta_4,r d_{ii} I + \beta_5,r d_{ii} D
\tag{16}
\]

\[
A_{i,j} = \beta_4,r d_{ij} I + \beta_5,r d_{ij} D, \quad \text{when } i \neq j,
\tag{17}
\]
Results

Comparison of Multi-domain bivariate spectral local linearisation solution for the skin friction against the series solution for large $\zeta$.

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$f''(0, \zeta)$</th>
<th>Series Solution for large $\zeta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.3088214</td>
<td>0.3088214</td>
</tr>
<tr>
<td>10</td>
<td>0.1547399</td>
<td>0.1547399</td>
</tr>
<tr>
<td>15</td>
<td>0.1031717</td>
<td>0.1031717</td>
</tr>
<tr>
<td>20</td>
<td>0.0773803</td>
<td>0.0773803</td>
</tr>
<tr>
<td>25</td>
<td>0.0619045</td>
<td>0.0619045</td>
</tr>
<tr>
<td>30</td>
<td>0.0515872</td>
<td>0.0515872</td>
</tr>
<tr>
<td>35</td>
<td>0.0442176</td>
<td>0.0442176</td>
</tr>
<tr>
<td>40</td>
<td>0.0386905</td>
<td>0.0386905</td>
</tr>
</tbody>
</table>

Table: $N_x = 60$, $N_t = 5$, $p = 20$
Results

Comparison of Multi-domain bivariate spectral local linearisation solution for the Sherwood number against the series solution for large $\zeta$

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$-h'(0, \zeta)$</th>
<th>Series Solution for large $\zeta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3.0018658</td>
<td>3.0018658</td>
</tr>
<tr>
<td>10</td>
<td>6.0002332</td>
<td>6.0002332</td>
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<tr>
<td>15</td>
<td>9.0000691</td>
<td>9.0000691</td>
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<tr>
<td>20</td>
<td>12.0000292</td>
<td>12.0000292</td>
</tr>
<tr>
<td>25</td>
<td>15.0000149</td>
<td>15.0000149</td>
</tr>
<tr>
<td>30</td>
<td>18.0000086</td>
<td>18.0000086</td>
</tr>
<tr>
<td>35</td>
<td>21.0000054</td>
<td>21.0000054</td>
</tr>
<tr>
<td>40</td>
<td>24.0000036</td>
<td>24.0000036</td>
</tr>
</tbody>
</table>

Table: $N_x = 60$, $N_t = 5$, $p = 20$
Results

- Comparison of Multi-domain bivariate spectral local linearisation solution for the **Nusselt number** against the series solution for large $\zeta$

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$-g'(0, \zeta)$ (MD-BSLLM)</th>
<th>$-g'(0, \zeta)$ (Series Solution for large $\zeta$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3.5018961</td>
<td>3.5018961</td>
</tr>
<tr>
<td>10</td>
<td>7.0002370</td>
<td>7.0002370</td>
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<td>15</td>
<td>10.5000702</td>
<td>10.5000702</td>
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<td>20</td>
<td>14.0000296</td>
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<td>25</td>
<td>17.5000152</td>
<td>17.5000152</td>
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<td>35</td>
<td>24.5000055</td>
<td>24.5000055</td>
</tr>
<tr>
<td>40</td>
<td>28.0000037</td>
<td>28.0000037</td>
</tr>
</tbody>
</table>

**Table:** $N_x = 60$, $N_t = 5$, $P = 20$
Conclusion

- The MD-BSLLM method can be used to solve non-linear non-similar boundary layer equations over a large time domain.

- We were able to validate the results using a series solution approach.

Future Research Direction

- Solve different types of NPDEs arising from Physics, Mathematical Biology, etc.


Thank you ...