TURBULENT AXISYMMETRIC FREE JET

- D P MASON

UNIVERSITY OF THE WITWATERSRAND

- D L HILL AND M MATTHEWS

UNIVERSITY OF WESTERN AUSTRALIA

OUTLINE

- THE REYNOLDS STRESSES
- PRANDTL'S MIXING LENGTH MODEL
- AXISYMMETRIC TURBULENT FREE JET
- ELEMENTARY CONSERVATION LAW
- ASSOCIATED LIE POINT SYMMETRY
- INVARIANT SOLUTION
- NUMERICAL SOLUTION
- CONCLUSIONS

- THE REYNOLDS STRESSES

x - COMPONENT OF NAVIER - STOKES EQUATION

$$\rho\left[\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z}\right] = -\frac{\partial p}{\partial x} + \mu\nabla^2 V_x$$

$$\underset{\substack{\uparrow\\ \text{MEAN}}}{V_x = \bar{V}_x} + \underset{\substack{\nwarrow\\ \text{FLUCTUATION}}}{V_x'}\ , \quad V_y = \bar{V}_y + V_y'\ , \quad V_z = \bar{V}_z + V_z'\ , \quad p = \bar{p} + p'$$

TAKE TIME AVERAGE

$$\overline{V_x'} = 0\ , \qquad \overline{V_x' V_y'} \neq 0$$

$$\rho\left[\bar{V}_x \frac{\partial \bar{V}_x}{\partial x} + \bar{V}_y \frac{\partial \bar{V}_x}{\partial y} + \bar{V}_z \frac{\partial \bar{V}_x}{\partial z}\right]$$

$$= \frac{\partial}{\partial x}\left[-p + \mu\left(\frac{\partial \bar{V}_x}{\partial x} + \frac{\partial \bar{V}_x}{\partial x}\right) - \rho\overline{V_x' V_x'}\right]$$

$$+ \frac{\partial}{\partial y}\left[+\mu\left(\frac{\partial \bar{V}_y}{\partial x} + \frac{\partial \bar{V}_x}{\partial y}\right) - \rho\overline{V_y' V_x'}\right]$$

$$+ \frac{\partial}{\partial z}\left[+\mu\left(\frac{\partial \bar{V}_z}{\partial x} + \frac{\partial \bar{V}_x}{\partial z}\right) - \rho\overline{V_z' V_x'}\right]$$

THE REYNOLDS STRESSES: $\tau_{ik}(T) = -\rho\overline{V_i' V_k'}$

BOUSSINESQUE (1877): $-\rho\overline{V_i' V_k'} = \mu_T\left(\frac{\partial \bar{V}_i}{\partial x_k} + \frac{\partial \bar{V}_k}{\partial x_i}\right)$

μ_T = EDDY VISCOSITY

$\nu_T = \frac{\mu_T}{\rho}$ = EDDY KINEMATIC VISCOSITY

ν_T DESCRIBES EFFECT OF FLUCTUATIONS ON MEAN FLOW

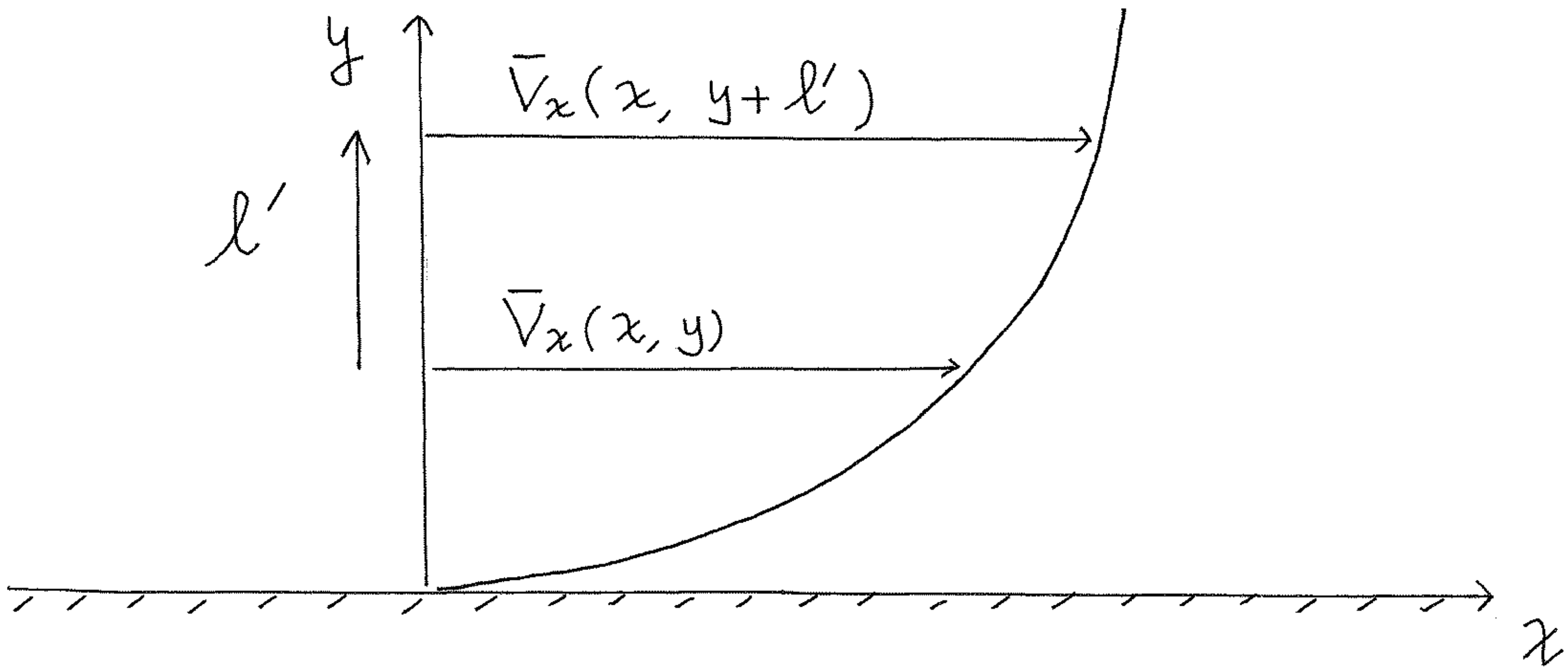
ν_T PROPERTY OF <u>FLOW</u> (NOT OF FLUID)

$$\nu_T = \nu_T\left(x, y, z, \bar{V}_i, \frac{\partial \bar{V}_i}{\partial x_k}, \dots\right)$$

$\nu + \nu_T$ = EFFECTIVE KINEMATIC VISCOSITY

$$\nu_T \gg \nu$$

- PRANDTL'S MIXING LENGTH MODEL



$$V_x'(x,y) = \bar{V}_x(x, y+\ell') - \bar{V}_x(x,y) = \ell' \frac{\partial \bar{V}_x}{\partial y}$$

ASSUME $\quad V_y'(x,y) = -c\, V_x'(x,y) \quad (c>0)$

$$-\rho \overline{V_x' V_y'} = \rho c\, \overline{\ell' \ell'} \left(\frac{\partial \bar{V}_x}{\partial y}\right)^2$$

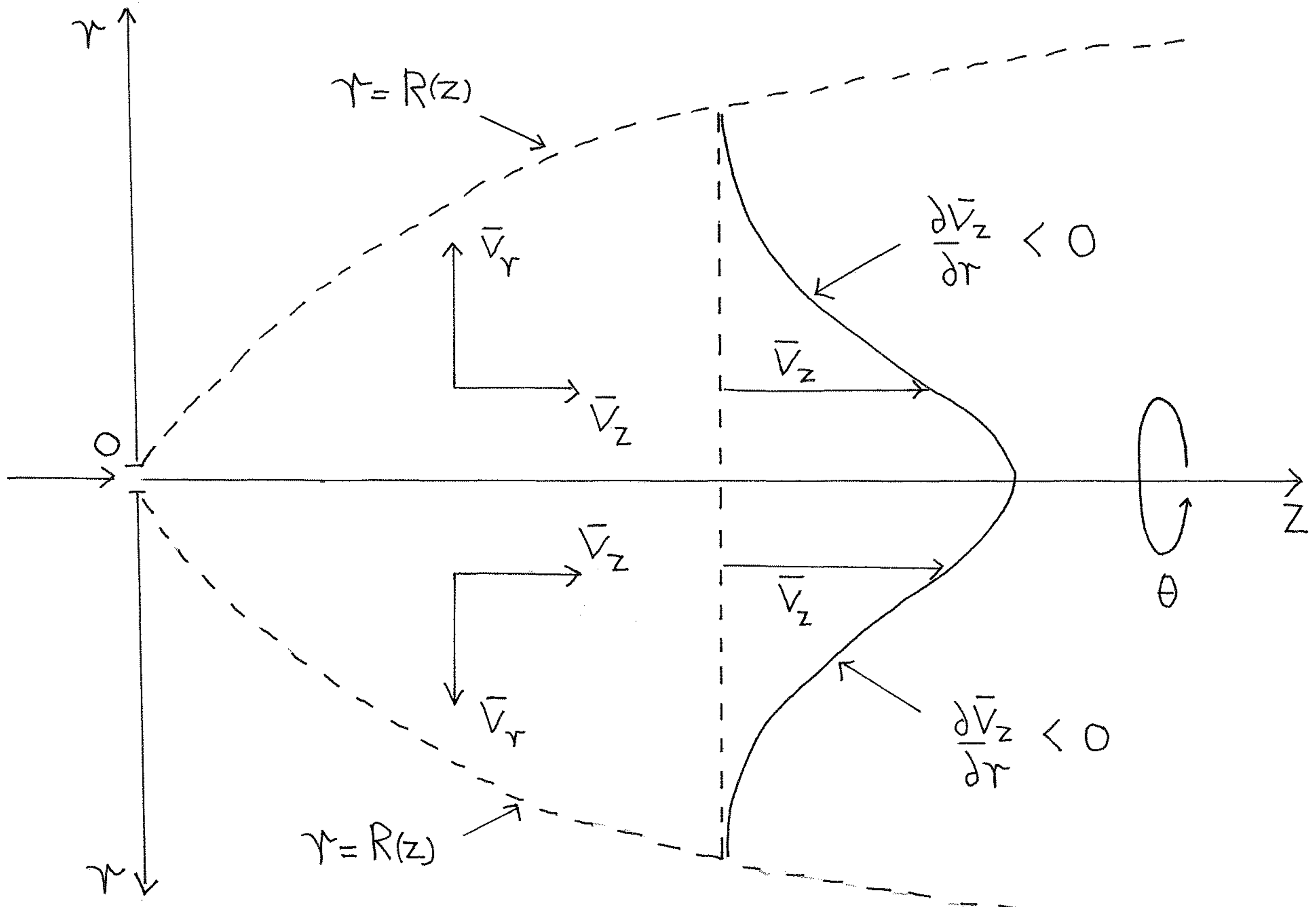
$$-\rho\,\overline{V_x' V_y'} = \rho \ell^2 \left|\frac{\partial \bar{V}_x}{\partial y}\right| \frac{\partial \bar{V}_x}{\partial y} \qquad \ell^2 = c\,\overline{\ell' \ell'}$$

$$\mu_T \frac{\partial \bar{V}_x}{\partial y} = \rho \ell^2 \left|\frac{\partial \bar{V}_x}{\partial y}\right| \frac{\partial \bar{V}_x}{\partial y}$$

$$\boxed{\nu_T = \frac{\mu_T}{\rho} = \ell^2 \left|\frac{\partial \bar{V}_x}{\partial y}\right|}$$

PRANDTL'S MIXING LENGTH $\quad \ell = \ell(x, y)$

- TURBULENT AXISYMMETRIC FREE JET



PRANDTL'S BOUNDARY LAYER EQUATIONS

$$\bar{V}_r \frac{\partial \bar{V}_z}{\partial r} + \bar{V}_z \frac{\partial \bar{V}_r}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left[E\left(z, \frac{\partial \bar{V}_z}{\partial r}\right) r \frac{\partial \bar{V}_z}{\partial r} \right]$$

$$\frac{\partial}{\partial r}(r \bar{V}_r) + \frac{\partial}{\partial z}(r \bar{V}_z) = 0$$

$$E\left(z, \frac{\partial \bar{V}_z}{\partial r}\right) = \nu + l^2(z) \left| \frac{\partial \bar{V}_z}{\partial r} \right|$$

$$= \nu - l^2(z) \frac{\partial \bar{V}_z}{\partial r} \qquad \left(\frac{\partial \bar{V}_z}{\partial r} < 0 \right)$$

MIXING LENGTH $l(z)$ NOT SPECIFIED

$l(z)$ OBTAINED FROM INVARIANT SOLUTION

COMPARISON WITH POWER-LAW FLUID JET

$$\bar{V}_r \frac{\partial \bar{V}_z}{\partial r} + \bar{V}_z \frac{\partial \bar{V}_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu - \ell^2(z)\frac{\partial \bar{V}_z}{\partial r}\right) r \frac{\partial \bar{V}_z}{\partial r}\right]$$

$$\bar{V}_r \frac{\partial \bar{V}_z}{\partial r} + \bar{V}_z \frac{\partial \bar{V}_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[E_o\left(-\frac{\partial V_z}{\partial r}\right)^{n-1} r \frac{\partial V_z}{\partial r}\right]$$

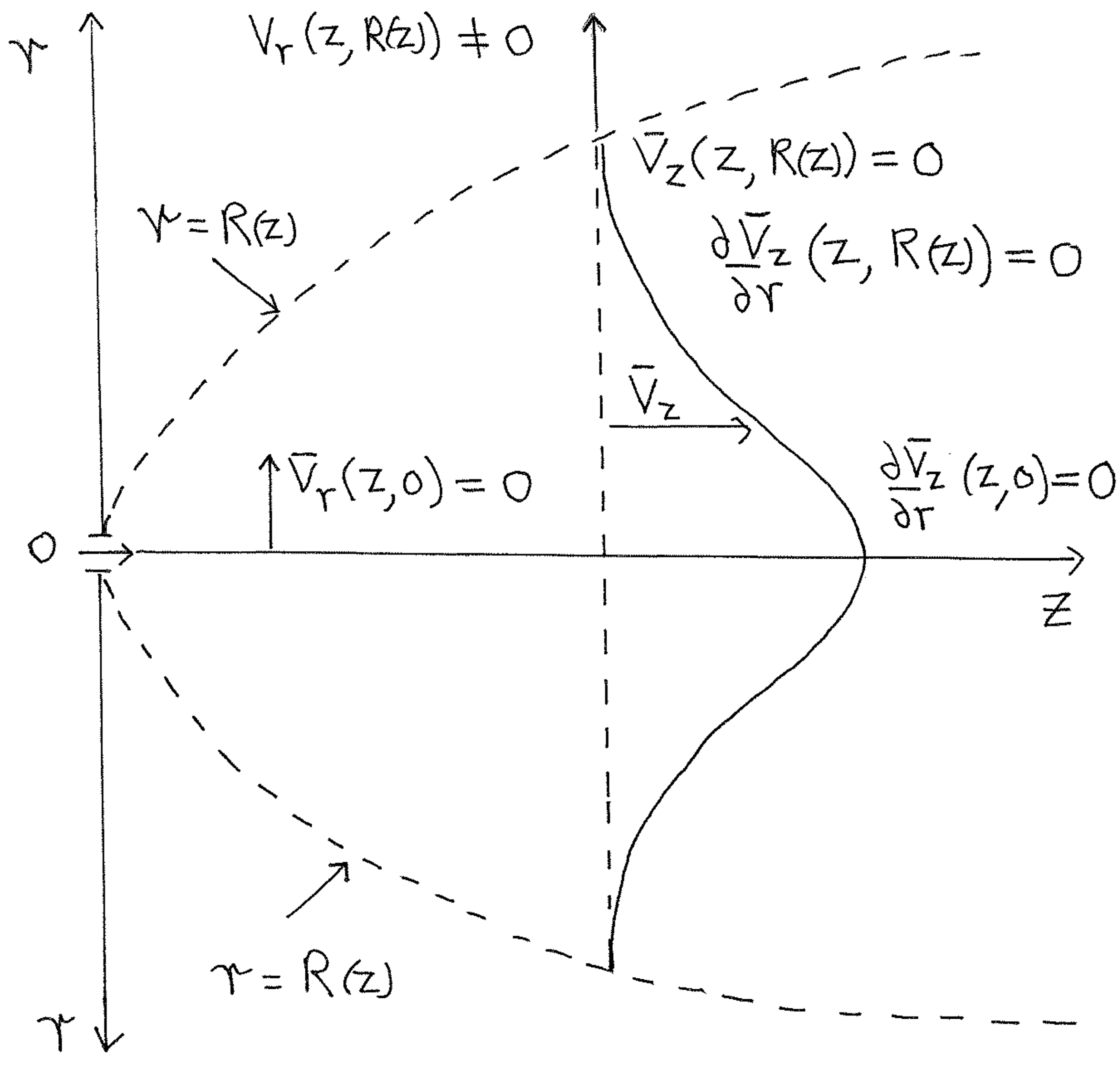
AXISYMMETRIC JET REDUCES TO POWER-LAW JET WHEN

$\nu = 0$ (NEGLECT KINEMATIC VISCOSITY)

$\ell(z) = \ell_o$ (MIXING LENGTH CONSTANT)

$n = 2$ (SHEAR <u>THICKENING</u> FLUID)

- BOUNDARY CONDITIONS



$r = 0$: $\bar{V}_r(z, 0) = 0$

$r = 0$: $\frac{\partial \bar{V}_z}{\partial r}(z, 0) = 0$

$r = R(z)$: $\bar{V}_z(z, R(z)) = 0$

$r = R(z)$: $\frac{\partial \bar{V}_z}{\partial r}(z, R(z)) = 0$

HOMOGENEOUS BOUNDARY CONDITIONS

NOTE

$r = R(z)$: $\bar{V}_r(z, R(z)) \neq 0$

- CONSERVATION LAWS

STREAM FUNCTION: $\bar{V}_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$, $\bar{V}_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$

$$-\frac{\partial \psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial^2 \psi}{\partial z \partial r} = \frac{\partial}{\partial r}\left[r\left(\nu - \ell^2(z)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right]$$

INDEPENDENT VARIABLES: $z, r, \psi, \psi_z, \psi_r, \ldots$

$$D_z T^1 + D_r T^2 \Big|_{PDE} = 0$$

$$D_z = \frac{\partial}{\partial z} + \psi_z \frac{\partial}{\partial \psi} + \psi_{zz}\frac{\partial}{\partial \psi_z} + \psi_{rz}\frac{\partial}{\partial \psi_r} + \ldots$$

$$D_r = \frac{\partial}{\partial r} + \psi_r \frac{\partial}{\partial \psi} + \psi_{zr}\frac{\partial}{\partial \psi_z} + \psi_{rr}\frac{\partial}{\partial \psi_r} + \ldots$$

- ELEMENTARY CONSERVED VECTOR

$$T^1 = \frac{1}{r}\psi_r^2$$

$$T^2 = -\frac{1}{r}\psi_z\psi_r - \nu\left(\psi_{rr} - \frac{1}{r}\psi_r\right) + \ell^2(z)\left(\psi_{rr}^2 - \frac{2}{r}\psi_r\psi_{rr} + \frac{1}{r^2}\psi_r^2\right)$$

- CONSERVED QUANTITY

INTEGRATE ELEMENTARY CONSERVATION LAW ACROSS JET

$$J = 2\pi\rho\int_0^{R(z)} \frac{1}{r}\left(\frac{\partial\psi}{\partial r}(r,z)\right)^2 dr = \text{CONSTANT}$$

(INDEPENDENT OF z)

- ASSOCIATED LIE POINT SYMMETRY

LIE POINT SYMMETRY

$$X = \xi^1(z,r,\psi)\frac{\partial}{\partial z} + \xi^2(z,r,\psi)\frac{\partial}{\partial r} + \eta(z,r,\psi)\frac{\partial}{\partial \psi}$$

IS ASSOCIATED WITH CONSERVED VECTOR $T=(T^1,T^2)$ PROVIDED (KARA AND MAHOMED, 2000)

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0$$

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0$$

$$D_1 = D_z, \quad D_2 = D_r$$

ELEMENTARY CONSERVED VECTOR

$$T^1 = \frac{1}{r}\psi_r^2$$

$$T^2 = -\frac{1}{r}\psi_z\psi_r - \nu\left(\psi_{rr} - \frac{1}{r}\psi_r\right) + \ell^2(z)\left(\psi_{rr}^2 - \frac{2}{r}\psi_r\psi_{rr} + \frac{1}{r^2}\psi_r^2\right)$$

RESULT:

$$X = \xi^1(z)\frac{\partial}{\partial z} + c_2 r\frac{\partial}{\partial r} + (c_3 + c_2\psi)\frac{\partial}{\partial \psi}$$

PROVIDED

- $\nu\left(\dfrac{d\xi^1}{dz} - c_2\right) = 0$

- $2\xi^1(z)\dfrac{d\ell}{dz} + \left(\dfrac{d\xi^1}{dz} - 3c_2\right)\ell(z) = 0$

(i) $\nu \neq 0$

$$\xi'(z) = c_1 + c_2 z$$

$$\ell(z) = \ell_0 (c_1 + c_2 z)$$

$$X = (c_1 + c_2 z)\frac{\partial}{\partial z} + c_2 r \frac{\partial}{\partial r} + (c_3 + c_2 \psi)\frac{\partial}{\partial \psi}$$

X SAME AS FOR LAMINAR FLOW

(ii) $\nu = 0$ $\quad \xi'(z)$ ARBITRARY

$$\ell(z) = \ell(0)\left(\frac{\xi'(0)}{\xi'(z)}\right)^{\frac{1}{2}} \exp\left[\frac{3}{2} c_2 \int_0^z \frac{dz}{\xi'(z)}\right]$$

$$X = \xi'(z) + c_2 r \frac{\partial}{\partial r} + (c_3 + c_2 \psi)\frac{\partial}{\partial \psi}$$

- ALTHOUGH $\nu \ll \nu_T$ IT CANNOT BE NEGLECTED

  $\nu \neq 0$ DETERMINES: $\xi^1(z)$ IN LIE POINT SYMMETRY X

  MIXING LENGTH $\ell(z)$

- WHEN $\nu = 0$, $\xi^1(z)$ IS ARBITRARY

  $\ell(z)$ DEPENDS ON ARBITRARY $\xi^1(z)$

- AFTER $\xi^1(z)$ AND $\ell(z)$ HAVE BEEN FOUND THE APPROXIMATION $\nu = 0$ CAN BE CONSIDERED TO LOOK FOR APPROXIMATE ANALYTICAL SOLUTION

- INVARIANT SOLUTION ($\nu \neq 0$)

$$X = (c_1 + z)\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} + (c_3 + \psi)\frac{\partial}{\partial \psi}$$

$\psi = \Psi(z,r)$ IS AN INVARIANT SOLUTION GENERATED BY X PROVIDED

$$X\left(\psi - \Psi(z,r)\right)\Big|_{\psi=\Psi} = 0$$

RESULT :

$$\psi(z,r) = (c_1 + z)\,F(\xi) - c_3 \qquad (c_3 = 0)$$

$$\xi = \frac{r}{c_1 + z}$$

$$\frac{d}{d\xi}\left[\left(\nu - \ell_0^2 \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right)\xi\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right] + \frac{d}{d\xi}\left[\frac{F(\xi)}{\xi}\frac{dF}{d\xi}\right] = 0$$

- CONSERVED QUANTITY

$$J = 2\pi\rho \int_0^{R(z)} \frac{1}{r}\left(\frac{\partial\psi}{\partial r}(z,r)\right)^2 dr = 2\pi\rho \int_0^{\frac{R(z)}{c_1+z}} \frac{1}{\xi}\left(\frac{dF}{d\xi}\right)^2 d\xi$$

= CONSTANT INDEPENDENT OF z

THUS:

$$\frac{R(z)}{c_1+z} = k, \qquad R(z) = k(c_1+z)$$

$$R(0) = 0 \qquad C_1 = 0$$

$$R(z) = kz \qquad (k \text{ TO BE DETERMINED})$$

$$l(z) = l_0 z \qquad (l_0 \text{ GIVEN})$$

IF k IS <u>FINITE</u> THEN AXISYMMETRIC JET IS <u>BOUNDED</u>

- BOUNDARY CONDITIONS

$$\bar{V}_r = \frac{1}{z}\left[\frac{dF}{d\xi} - \frac{F(\xi)}{\xi}\right], \qquad \bar{V}_z = \frac{1}{z}\frac{1}{\xi}\frac{dF}{d\xi}$$

$r=0$: $\bar{V}_r(z,0) = 0$ $\qquad$ $\left.\frac{dF}{d\xi} - \frac{F(\xi)}{\xi}\right|_{\xi=0} = 0$

$r=0$: $\frac{\partial \bar{V}_z}{\partial r}(z,0) = 0$ $\qquad$ $\left.\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right|_{\xi=0} = 0$

$r=kz$: $\bar{V}_z(z, kz) = 0$ $\qquad$ $\frac{dF}{d\xi}(k) = 0$

$r=kz$: $\begin{cases} \nu_T = 0 \\ \frac{\partial \bar{V}_z}{\partial r}(z, kz) = 0 \end{cases}$ $\qquad$ $\frac{d^2F}{d\xi^2}(k) = 0$

- DOUBLE REDUCTION THEOREM (SJOBERG, 2007)

$$\frac{d}{d\xi}\left[\left(\nu - l_0^2 \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right)\xi \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right] + \frac{d}{d\xi}\left[\frac{F(\xi)}{\xi}\frac{dF}{d\xi}\right] = 0$$

INTEGRATE ONCE. IMPOSE BOUNDARY CONDITIONS

$$\left[\nu - l_0^2 \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right]\xi \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right) + \frac{F(\xi)}{\xi}\frac{dF}{d\xi} = 0$$

DOUBLE REDUCTION OF PDE TO ODE USING ASSOCIATED LIE POINT SYMMETRY

- REDUCTION OF PDE TO ODE
- REDUCTION OF ODE OF ORDER 3 TO ODE OF ORDER 2

- SOLVE FOR $F(\xi)$ AND k

$$\left[\nu - l_0^2 \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right]\xi \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right) + \frac{F(\xi)}{\xi}\frac{dF}{d\xi} = 0$$

BOUNDARY CONDITIONS

$$\left.\frac{dF}{d\xi} - \frac{F(\xi)}{\xi}\right|_{\xi=0} = 0, \quad \left.\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right|_{\xi=0} = 0, \quad \frac{dF}{d\xi}(k) = 0, \quad \frac{d^2}{d\xi^2}F(k) = 0$$

CONSERVED QUANTITY

$$J = 2\pi\rho \int_0^k \frac{1}{\xi}\left(\frac{dF}{d\xi}\right)^2 d\xi \qquad (J \text{ GIVEN})$$

REMAINING QUANTITIES

$$R(z) = kz, \qquad l(z) = l_0 z \qquad (l_0 \text{ GIVEN})$$

$$V_z = \frac{1}{z}\frac{1}{\xi}\frac{dF}{d\xi}, \quad V_r = \frac{1}{z}\left[\frac{dF}{d\xi} - \frac{F(\xi)}{\xi}\right], \quad X = z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} + \psi\frac{\partial}{\partial \psi}$$

- NUMERICAL SOLUTION

$$F = \nu \bar{F}, \qquad \lambda = l_0^2$$

SYSTEM SINGULAR AT $\xi = 0$.

CONSIDER

$$\bar{F}(\xi) = A\xi^n \qquad \text{AS} \quad \xi \to 0.$$

ODE: $$\left[1 - \lambda \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d\bar{F}}{d\xi}\right)\right] \xi \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d\bar{F}}{d\xi}\right) + \frac{\bar{F}}{\xi}\frac{d\bar{F}}{d\xi} = 0$$

$$(n-2)\xi^{n-2} - \lambda n (n-2)^2 A \xi^{2n-5} + A\xi^{2n-2} = 0 \quad \text{AS } \xi \to 0$$

REQUIRES $\quad n = 2 \quad$ OR $\quad n > \frac{5}{2}$.

BOUNDARY CONDITIONS

$$\left. \frac{dF}{d\xi} - \frac{F(\xi)}{\xi} \right|_{\xi=0} = 0 \quad \text{AND} \quad \left. \frac{d}{d\xi}\left(\frac{1}{\xi} \frac{dF}{d\xi} \right) \right|_{\xi=0} = 0$$

$$(n-1) A \xi^{n-1} = 0 \quad \text{AND} \quad n(n-2) A \xi^{n-3} = 0 \quad \text{AS } \xi \to 0$$

REQUIRES $n=2$ OR $n>3$.

CONSIDER $n=2$.

NUMERICAL PROBLEM

$$\frac{d^2\bar{F}}{d\xi^2} = \frac{1}{\xi}\frac{d\bar{F}}{d\xi} + \frac{1}{2\lambda}\left[\xi - \left(\xi^2 + 4\lambda\bar{F}\frac{d\bar{F}}{d\xi}\right)^{\frac{1}{2}}\right] \qquad (\lambda \text{ GIVEN})$$

SUBJECT TO

$$\bar{F}(\delta) = A\delta^2, \quad \frac{d\bar{F}}{d\xi}(\delta) = 2A\delta$$

$$\frac{J}{2\pi\rho\nu^2} = \int_0^k \frac{1}{\xi}\left(\frac{d\bar{F}}{d\xi}\right)^2 d\xi \qquad \left(\frac{J}{2\pi\rho\nu^2} \text{ GIVEN}\right)$$

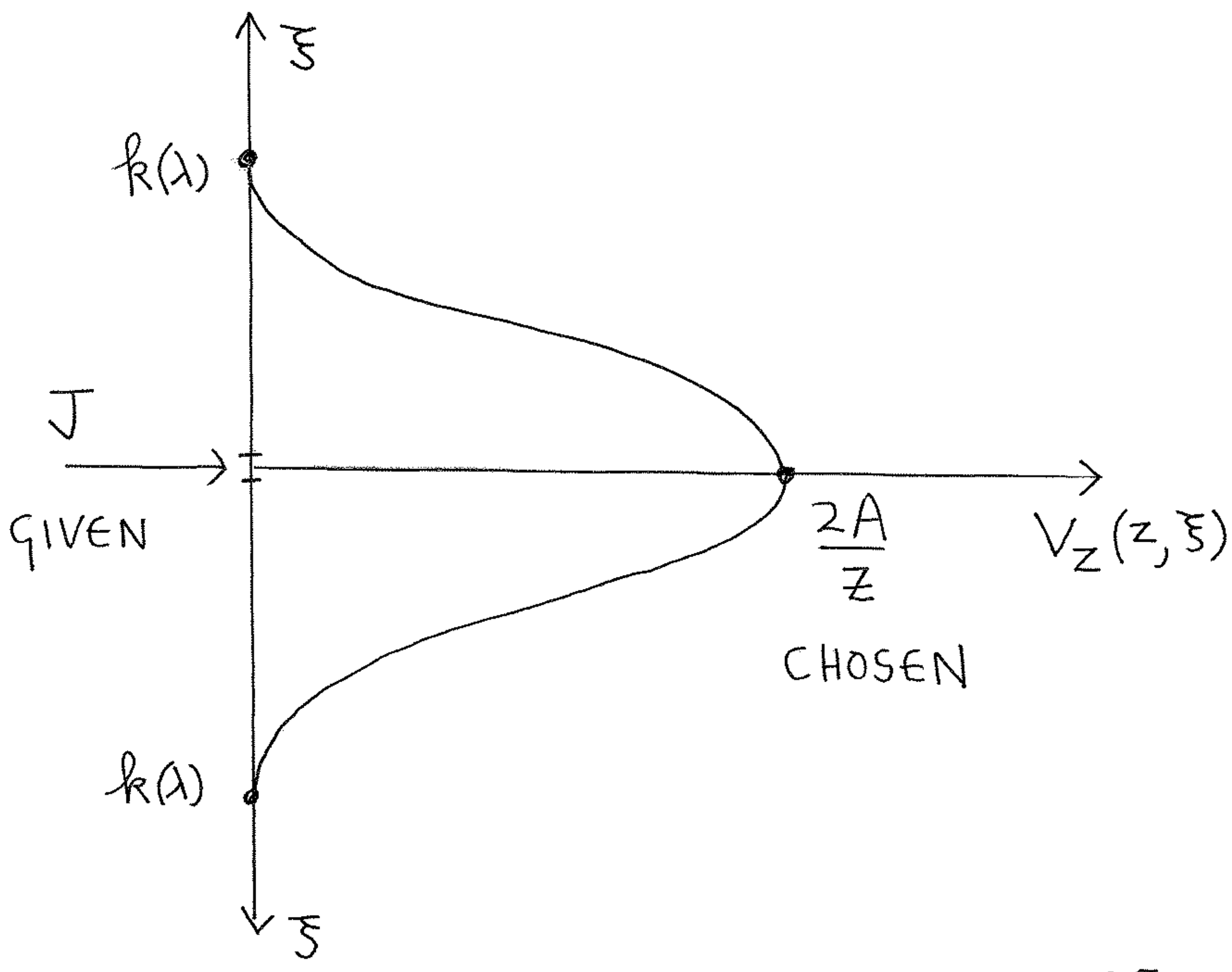
$$\left(\frac{d\bar{F}}{d\xi}(k) = 0, \quad \frac{d^2\bar{F}}{d\xi^2}(k) = 0 \right)$$

FOR A GIVEN VALUE OF A AND SPECIFIED VALUE OF $\delta \ll 1$ THE SOLUTION IS OBTAINED FOR k AND $\bar{F}(\xi)$ FOR $\xi \geq \delta$.

PHYSICAL SIGNIFICANCE OF A

$$V_z(z,\xi) = \frac{1}{z}\frac{1}{\xi}\frac{dF}{d\xi}, \qquad V_z(z,\delta) = \frac{2A}{z} \qquad (F(\delta) = A\delta^2)$$

$= \text{VELOCITY ON AXIS}$



$$\frac{J}{2\pi\rho\nu^2} = \int_0^k \frac{1}{\xi}\frac{dF}{d\xi}\,d\xi$$

CONCLUSIONS

- ALTHOUGH $\nu \ll \nu_T$ IT CANNOT BE NEGLECTED

  ν DETERMINES $\xi'(z)$ IN X AND MIXING LENGTH $\ell(z)$

- FOR AN INVARIANT SOLUTION THE MIXING LENGTH MUST BE A POWER-LAW AND INCREASE LINEARLY ALONG THE JET

$$\ell(z) = \ell_0 z .$$