# Beyond integrability: the far-reaching consequences of thinking about boundary conditions

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# Introduction

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**Question**: Is it possible to extend the applicability of the IST to **boundary value problems**?

**Answer**: *Do we really understand* **linear** *BVPs?*
Novel point of view for deriving integral transforms - the **Unified Transform** (UT) (*Fokas*)
$\longrightarrow$ results in linear, spectral and numerical theory and applications

# The most important integral transform: Fourier transform

**Given** $u(x)$ a smooth, decaying function on $\mathbb{R}$, the Fourier transform is the map

$$u(x) \quad \rightarrow \quad \hat{u}(\lambda, t) = \int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda x} u(x) dx, \qquad \lambda \in \mathbb{R}.$$

**Given** $\hat{u}(\lambda)$, $\lambda \in \mathbb{R}$, with sufficient decay as $\lambda \rightarrow \infty$, we can use the inverse transform to represent $u(x)$:

$$\hat{u}(\lambda) \quad \rightarrow \quad u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x} \hat{u}(\lambda) d\lambda, \qquad x \in \mathbb{R}.$$

# Linear Initial Value Problems, $x \in \mathbb{R}$

$$u_t + u_{xxx} = 0, \qquad u(x,0) = u_0(x) \; smooth, \; decaying \; at \; \pm\infty$$

(*linear part of the KdV equation*: $u_t + u_{xxx} + uu_x = 0$)

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(*linear part of the KdV equation*: $u_t + u_{xxx} + uu_x = 0$)

on $\mathbb{R} \times (0, T)$: use Fourier Transform:

$$\hat{u}_t(\lambda, t) + (i\lambda)^3 \hat{u}(\lambda, t) = 0, \quad \lambda \in \mathbb{R}$$

*solution* : $$u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{i\lambda x + i\lambda^3 t} \hat{u}_0(\lambda) d\lambda.$$

# ...with a boundary - do we need another transform?

on $(0,\infty)\times(0,T)$:

$$u_t + u_{xxx} = 0, \qquad u(x,0) = u_0(x), \quad u(0,t) = u_1(t)$$

Fourier transform the equation:

$$\hat{u}_t(\lambda,t) + (i\lambda)^3\hat{u}(\lambda,t) = u_{xx}(0,t) + i\lambda u_x(0,t) - \lambda^2 u(0,t)$$

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"*solution*" :

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x + i\lambda^3 t} \hat{u}_0(\lambda) d\lambda +$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x + i\lambda^3 t} \left[ \int_0^t \mathrm{e}^{-i\lambda^3 s} \left( u_{xx}(0,s) + i\lambda u_x(0,s) - \lambda^2 u_1(s) \right) ds \right] d\lambda.$$

(Note: *could use Laplace transform*)

# 2-point boundary value problems

$$u_t = u_{xx}, \quad u(x,0) = u_0(x), \ \ in\ [0,1]\times(0,T), \quad \text{2 boundary cond's}$$

use separation of var's and eigenfunctions of

$$S = \frac{d^2}{dx^2}\ on\ \mathcal{D} = \{f \in C^\infty([0,1]) : f\ satisfies\ the\ bc's\} \subset L^2[0,1]$$

$S$ is selfadjoint and we can compute its eigenvalues $\lambda_n$ and eigenfunctions $\phi_n$ (depend on bc's), and

$$u(x,t) = \sum_n (u_0,\phi_n)\mathrm{e}^{-\lambda_n^2 t}\phi_n(x) \left( e.g. = \sum_j \hat{u}_0(n)\mathrm{e}^{-(\pi n)^2 t}\sin \pi n x \right)$$

# 2-point boundary value problems

on $[0,1] \times (0, T)$:

$$u_t = u_{xxx}, \qquad u(x,0) = u_0(x), \qquad \text{3 b'dary cond's (?)}$$

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Separate variables, and use eigenfunctions of

$$S = i\frac{d^3}{dx^3} \ on \ \mathcal{D} = \{f \in C^\infty([0, 1]) : f \ satisfies \ 3 \ bc's\} \subset L^2[0, 1]?$$

$S$ is not generally selfadjoint (because of BC), but has infinitely many real eigenvalues $\lambda_n$, and associated eigenfunctions $\{\phi_n(x)\}$

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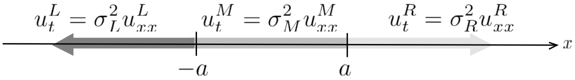
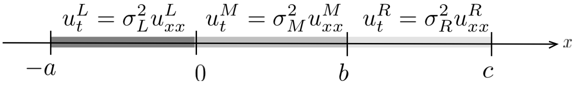
$$\rightarrow \quad u(x,t) = \sum_n (u_0, \phi_n) \mathrm{e}^{i\lambda_n^3 t} \phi_n(x)?$$

# Evolution problems with time-dependent boundaries, multiple point boundary conditions or interfaces




*The heat conduction problem for a single rod of length 2a between two semi-infinite rods*




*The heat equation for three finite layers*

....given initial, boundary and interface conditions

(More generally, heat distribution on a graph)

# Another type of linear problems: elliptic PDEs

$$\Delta u + 4\beta^2 u = 0, \quad \mathbf{x} \in \Omega, \qquad u = f \text{ on } \partial\Omega$$

where $\Omega \subset \mathbb{R}^d$ is a simply connected, convex domain,

$\beta = 0$ Laplace

$\beta \in i\mathbb{R}$ Modified Helmholtz

$\beta \in \mathbb{R}$ Helmholtz

(Largely open) questions:

- Effective closed form solution representation
- Characterization of the spectral structure
- Generalization to non-convex or multiply-connected domains

*Motivation*: solving nonlinear integrable equations of elliptic type, e.g. $u_{xx} + u_{yy} + \sin u = 0, \ x, y \in \Omega$

# Nonlinear integrable evolution PDEs

**Integrable PDEs**: $\sim$ "PDEs with infinitely many symmetries"

*NLS, KdV, sine-Gordon, elliptic sine-Gordon,...*

**IST**: $\sim$ a nonlinear integral (Fourier) transform

**The key ingredients**

(1): Lax pairs formulation of (integrable) PDEs
(Lax, Zakharov, Shabat 1970's)

(2): Riemann-Hilbert formulation of integral transforms
(Fokas-Gelfand 1994)

# Riemann-Hilbert (RH) problem

*The reconstruction of a (sectionally) analytic function from the prescribed jump across a given curve*

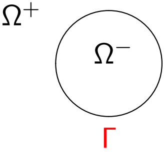
**Given**

(a) a contour $\Gamma \subset \mathbb{C}$ that divides $\mathbb{C}$ into two subdomains $\Omega^+$ and $\Omega^-$

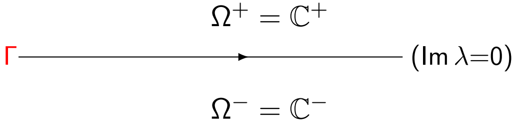
(b) a scalar/matrix valued function $G(\lambda),\ \lambda \in \Gamma$

**find** $H(z)$ analytic off $\Gamma$ (plus normalisation - e.g. $H \sim I$ at infinity):

$$H^+(\lambda) = H^-(\lambda)G(\lambda)\ (\lambda \in \Gamma); \qquad H^\pm(\lambda) = \lim_{z \to \Gamma^\pm} H(z)$$




*or*



# RH formulation of integral transforms

Example: The ODE

$$\mu_x(x,\lambda) - i\lambda\mu(x,\lambda) = u(x), \quad \lambda \in \mathbb{C}$$

encodes the Fourier transform

direct transform: via solving the ODE for $\mu(x,\lambda)$ *bounded in $\lambda$*
inverse transform: via solving a RH problem

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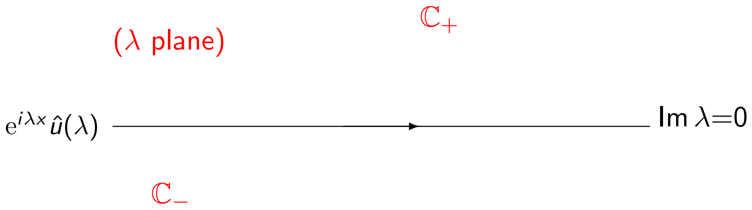
**Given** $u(x)$ (smooth and decaying), solutions $\mu^+$ and $\mu^-$ bounded (wrt $\lambda$) in $\mathbb{C}^+$ and $\mathbb{C}^-$ are

$$\mu^+ = \int_{-\infty}^{x} \mathrm{e}^{i\lambda(x-y)} u(y)dy,\ \lambda \in \mathbb{C}^+; \quad \mu^- = \int_{\infty}^{x} \mathrm{e}^{i\lambda(x-y)} u(y)dy,\ \lambda \in \mathbb{C}^-$$

$$\Longrightarrow \textit{for } \lambda \in \mathbb{R} \qquad (\mu^+ - \mu^-)(\lambda) = \mathrm{e}^{i\lambda x}\hat{u}(\lambda) \qquad \textbf{DIRECT}$$

# Fourier inversion theorem




**Given** $\hat{u}(\lambda)$, $\lambda \in \mathbb{R}$, a function $\mu$ analytic everywhere in $\mathbb{C}$ except the real axis is the solution of a RH problem (via *Plemelj formula*):

$$\mu(\lambda, x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\zeta x}\hat{u}(\zeta)}{\zeta - \lambda} d\zeta$$

$$\Rightarrow u(x) = \mu_x - i\lambda\mu = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\zeta x}\hat{u}(\zeta) d\zeta, \ x \in \mathbb{R} \qquad \textbf{INVERSE}$$

# Inverse Scattering Transform - a nonlinear FT

*The ODE* $\mu_x - i\lambda\mu = u(x)$ *encodes the Fourier transform*

Same idea, but use a matrix -valued ODE

$$M_x + i\lambda[\sigma_3, M] = UM, \quad M(x,\lambda) \text{ a } 2\times 2 \text{ matrix},$$

$$U = \begin{pmatrix} 0 & u(x) \\ \pm\bar{u}(x) & 0 \end{pmatrix}, \ \sigma_3 = diag(1,-1), \ [\sigma_3, M] = \sigma_3 M - M\sigma_3$$

**Direct problem**: given $U$ find $M$ solving the ODE above,

$$M \sim I \text{ as } \lambda \to \infty.$$

Then $M$ is analytic everywhere off $\mathbb{R}$, and one can compute its jump across the real line

# IST - the inverse transform

**Inverse problem**: Given the curve ($\mathbb{R}$) and the jump across it, find

$$M\text{:}\ \ M \sim I + O\left(\frac{1}{\lambda}\right) \text{ as } \lambda \to \infty$$

The matrix $M$ is the (unique) solution of the associated matrix-valued (multiplicative) Riemann-Hilbert problem on $\mathbb{R}$

$\Longrightarrow$ find $u$ as

$$u(x) = 2i \lim_{|\lambda| \to \infty} (\lambda M_{12}(x, \lambda)),$$

(linear case: $\mu \sim iu/\lambda$ as $\lambda \to \infty$)

**IST: a nonlinear Fourier transform**

# Lax pair formulation: the key to integrability of PDEs

Example: nonlinear Schrödinger equation

$$iu_t + u_{xx} - 2u|u|^2 = 0 \Longleftrightarrow M_{xt} = M_{tx}, \qquad U = \begin{pmatrix} 0 & u(x) \\ \overline{u(x)} & 0 \end{pmatrix}$$

$$\begin{cases} M_x + i\lambda[\sigma_3, M] = UM \\ M_t + 2i\lambda^2[\sigma_3, M] = (2\lambda U - iU_x\sigma_3 - i|u|^2\sigma_3)M \end{cases}$$

($M$ a $2\times 2$ matrix; $\sigma_3 = diag(1,-1)$; $[\sigma_3, M] = \sigma_3 M - M\sigma_3$)

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**Given this Lax pair formulation**:

- find M (bdd in $\lambda$) from 1st ODE + time evolution of $M$ is linear $\rightarrow$ **solve for** $M(\lambda; x, t)$

- then use RH to invert and **find an expression for** $u$:

$$u(x,t) = 2i \lim_{|\lambda|\to\infty} (\lambda M_{12}(\lambda; x, t))$$

# Back to linear PDEs

Linear (constant coefficient) PDE in two variables $\rightarrow$

**Lax pair formulation**

PDE as the *compatibility condition of a pair of linear ODEs*

Example: linear evolution problem

$$u_t + u_{xxx} = 0 \Longleftrightarrow \mu_{xt} = \mu_{tx} \quad \text{with} \quad \mu : \begin{cases} \mu_x - i\lambda\mu = u \\ \mu_t - i\lambda^3\mu = u_{xx} + i\lambda u_x - \lambda^2 u \end{cases}$$

**Main idea:** derive a transform pair (via RH) from this **system** of ODEs

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*equivalently, divergence form (classical for elliptic case)*

$$u_t + u_{xxx} = 0 \Longleftrightarrow [\mathrm{e}^{-i\lambda x - i\lambda^3 t} u]_t - [\mathrm{e}^{-i\lambda x - i\lambda^3 t}(u_{xx} + i\lambda u_x - \lambda^2 u)]_x = 0$$

# Spectral transforms+Lax pair $\rightarrow$ Unified Transform

**Unified (Fokas) Transform**:
system of ODEs (Lax pair) $\rightarrow$ RH problems $\rightarrow$ integral transform inversion

1: **Integral representation**
a *complex contour* representation - involves all boundary values of the solution

2: **Global relation** - *the heart of the matter for BVP*
compatibility condition *in spectral space* (the $\lambda$ space), involving transforms of all boundary values

# The role of the global relation

Invariance+analyticity properties of the global relation $\rightarrow$ representation only in terms of the given initial and boundary conditions

Always possible for

- linear evolution case
- linear elliptic case on symmetric domains
- "linearisable" nonlinear integrable case

effective, explicit integral representation of the solution

However, even if it is not possible to derive an explicit representation, the global relation yields an

additional set of relations and information about the solution

# Example when we can obtain an explicit final answer

$$u_t + u_{xxx} = 0, \qquad u(x,0) = u_0(x), \quad u(0,t) = u_1(t)$$

After exploiting the global relation and its analyticity/invariance properties:

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{i\lambda x + i\lambda^3 t}\hat{u}_0(\lambda)d\lambda$$

$$+\frac{1}{2\pi}\int_{\partial D_+} \mathrm{e}^{i\lambda x + i\lambda^3 t}\left[\omega\hat{u}_0(\omega\lambda) + \omega^2\hat{u}_0(\omega^2\lambda) - 3\lambda^2\tilde{u}_1(\lambda)\right]d\lambda$$

$$\tilde{u}_1(\lambda) = \int_0^T \mathrm{e}^{-i\lambda^3 s}u_1(s)ds, \qquad \omega = \mathrm{e}^{2\pi i/3}$$

$(\partial D^+ = \{\lambda \in \mathbb{C}^+ : Im(\lambda^3) = 0\})$

# Example - the global relation

$$\tilde{f}_2(\lambda^3) + i\lambda\tilde{f}_1(\lambda^3) - \lambda^2\tilde{f}_0(\lambda^3) = \hat{u}_0(\lambda) - \mathrm{e}^{-i\lambda^3 t}\hat{u}(\lambda, t), \quad \lambda \in \mathbb{C}^-$$

with

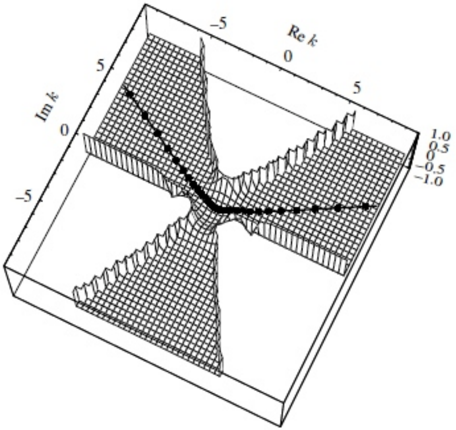
$$\tilde{f}_2(\lambda^3) + i\lambda\tilde{f}_1(\lambda^3) - \lambda^2\tilde{f}_0(\lambda^3) =$$

$$= \int_0^T \mathrm{e}^{-i\lambda^3 s} u_{xx}(0, s)ds + i\lambda \int_0^T \mathrm{e}^{-i\lambda^3 s} u_x(0, s)ds - \lambda^2 \int_0^T \mathrm{e}^{-i\lambda^3 s} u(0, s)ds$$

Important general facts:
(1) $\tilde{f}_i$ are invariant for any transformation that keeps $\lambda^3$ invariant
(2) terms involving $\hat{u}(\lambda, t)$ are analytic inside the domain of integration $D^+$

# Numerical application (Flyer, Fokas, Vetra, Shiels )

Evaluation of the integral representation via *contour deformation* (to contour in bold) and *uniform convergence* of the representation



# Heat interface problem (Deconinck, P, Shiels)

heat flow through two walls of semi-infinite width - explicit solution:

$$u^L(x,t) = \gamma^L + \frac{\sigma_R(\gamma^R - \gamma^L)}{\sigma_L + \sigma_R}\left(1 - erf\left(\frac{x}{2\sqrt{\sigma_L^2 t}}\right)\right)$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-(\sigma_L k)^2 t}\hat{u}_0^L(k)dk + \int_{\partial D^-}\frac{\sigma_R - \sigma_L}{2\pi(\sigma_L+\sigma_R)}e^{ikx-(\sigma_L k)^2 t}\hat{u}_0^L(-k)dk$$

$$-\int_{\partial D^-}\frac{\sigma_L}{\pi(\sigma_L+\sigma_R)}e^{ikx-(\sigma_L k)^2 t}\hat{u}_0^R(k\sigma_L/\sigma_R)dk,$$

$$u^R(x,t) = \gamma^R + \frac{\sigma_L(\gamma^L - \gamma^R)}{\sigma_L + \sigma_R}\left(1 - erf\left(\frac{x}{2\sqrt{\sigma_R^2 t}}\right)\right)$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-(\sigma_R k)^2 t}\hat{u}_0^R(k)dk + \int_{\partial D^+}\frac{\sigma_R - \sigma_L}{2\pi(\sigma_L+\sigma_R)}e^{ikx-(\sigma_R k)^2 t}\hat{u}_0^R(-k)dk$$

$$+\int_{\partial D^+}\frac{\sigma_R}{\pi(\sigma_L+\sigma_R)}e^{ikx-(\sigma_R k)^2 t}\hat{u}_0^L(k\sigma_R/\sigma_L)dk.$$

# Numerical evaluation of the solution - heat interface problem

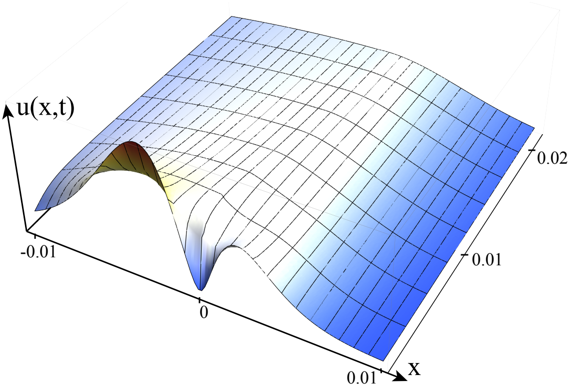



Figure: Results for the solution with $u_0^L(x) = x^2 e^{(25)^2 x}$, $u_0^R(x) = x^2 e^{-(30)^2 x}$ and $\sigma_L = .02$, $\sigma_R = .06$, $\gamma^L = \gamma^R = 0$, $t \in [0, 0.02]$ using the hybrid analytical-numerical method of Flyer

# Linear evolution problems - general result (*Fokas, P, Sung*)

$$\mathbf{u_t + iP(-i\partial_x)u = 0}, \qquad (P \text{ polynomial})$$

**Initial Value Problem:** $x \in \mathbb{R}$

$$u_0(x) \xrightarrow{\textit{direct}} \widehat{u_0}(\lambda) \xrightarrow{\textit{inverse}} u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{i\lambda x - iP(\lambda)t}\,\widehat{u_0}(\lambda)d\lambda$$

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**Boundary Value Problem:** $x \in I \subset \mathbb{R}^+$

$$\{u_0(x),\ f_j(t)\} \xrightarrow{\textit{direct}} \overset{+\ \textit{global relation}}{\{\widehat{u_0}(\lambda),\ \zeta(\lambda), \Delta(\lambda)\}} \xrightarrow{\textit{inverse}}$$

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{i\lambda x - iP(\lambda)t}\,\widehat{u_0}(\lambda)d\lambda + \int_{\partial D^{\pm}} \mathrm{e}^{i\lambda x - iP(\lambda)t}\,\frac{\zeta(\lambda)}{\Delta(\lambda)}d\lambda$$

$$\partial D^{\pm} = \{\lambda \in \mathbb{C}^{\pm} : Im(P(\lambda)) = 0\}$$

# Singularities in the RH data (P, Smith)

$$u_t + Su = 0,\ x \in I \qquad S = iP(-i\partial_x)(+\text{ b.c.})$$

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{i\lambda x - iP(\lambda)t}\widehat{u_0}(\lambda)d\lambda + \int_{\partial D} \mathrm{e}^{i\lambda x - iP(\lambda)t}\frac{\zeta(\lambda)}{\Delta(\lambda)}d\lambda$$

• $\widehat{u_0}(\lambda)$, $\zeta(\lambda)$, are *transforms* of the given initial and boundary conditions

• $\Delta(\lambda)$ is a determinant (arising in the solution of the global relation) whose zeros if they exist are (*essentially*) the discrete eigenvalues of $S$.

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Uniformly convergent representation, in contrast to not uniformly (slow) converging real integral/series representation

If the associated eigenfuctions form a basis (say the operator+bc is self-adjoint...), this representation is equivalent to the series one

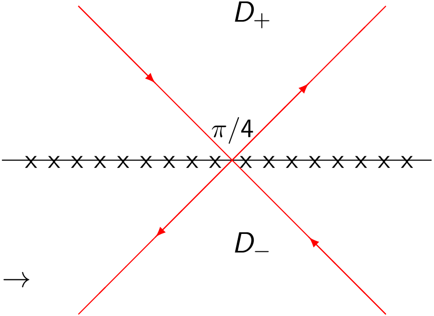
# Integral vs series representation

$$u_t = u_{xx}, \quad u(x,0) = u_0(x), \qquad u(0,t) = u(1,t) = 0$$

$$2\pi u(x,t) = \int_{\mathbb{R}} e^{i\lambda x - \lambda^2 t} \hat{u}_0(\lambda) d\lambda$$

$$- \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} \frac{e^{i\lambda}\hat{u}_0(-\lambda) - e^{-i\lambda}\hat{u}_0(\lambda)}{e^{-i\lambda} - e^{i\lambda}} d\lambda$$

$$- \int_{\partial D^-} e^{i\lambda(x-1) - \lambda^2 t} \frac{\hat{u}_0(\lambda) - \hat{u}_0(-\lambda)}{e^{-i\lambda} - e^{i\lambda}} d\lambda.$$




$\lambda_n = \pi n$ zeros of $\Delta(\lambda) = e^{-i\lambda} - e^{i\lambda}$

Using Cauchy+residue calculation $\rightarrow$

$$u(x,t) = \frac{2}{\pi} \sum_n e^{-\lambda_n^2 t} \sin(\lambda_n x) \hat{u}_0^s(\lambda_n) \qquad \textit{sine series}$$

# A very different example: the PDE $u_t = u_{xxx}$

$I = [0,1]$: zeros of $\Delta(\lambda)$ are an infinite set accumulating only at infinity; (asymptotic) location is given by general results in complex analysis , and **depends crucially on the boundary conditions** (P-Smith)

$$\textit{boundary conditions}: \quad u(0,t) = u(1,t) = 0, \quad u_x(0,t) = \beta u_x(1,t)$$

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- $-1 < \beta < 0$: the zeros are asymptotic to the integration contour $\to$ residue computation
- $\beta = 0$: the contour of integration cannot be deformed as far the asymptotic directions of the zeros
  $\Longrightarrow$ the underlying differential operator does not admit a Riesz basis of eigenfunctions

# More general spectral decomposition of differential operators: augmented eigenfunctions

$u_t + \partial_x^n u = 0$, $x \in [0,1]$ + initial and homogeneous boundary conditions

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{i\lambda x-(i\lambda)^n t}\,\widehat{u_0}(\lambda)d\lambda + \int_{\Gamma} \mathrm{e}^{i\lambda x-(i\lambda)^n t}\frac{\zeta(\lambda;u_0)}{\Delta(\lambda)}d\lambda$$

$F(f)(\lambda) = \frac{\zeta(\lambda;f)}{\Delta(\lambda)}$ is the family of **augmented eigenfunctions** of $S = \partial_x^n$ on $\{f \in C^\infty : f \text{ satisfies the boundary conditions }\} \subset L^2$

in the sense that

$$F(Sf)(\lambda) = \lambda^n F(f)(\lambda) + R(f)(\lambda): \quad \int_{\Gamma} \mathrm{e}^{i\lambda x}\frac{R(f)}{\lambda^n}d\lambda = 0.$$

# Elliptic problems in convex polygons (Ashton, Fokas, Kapaev, Spence)

$$u_{z\bar{z}} + \beta^2 u = 0, \quad z = x + iy \in \Omega \textit{ convex polygon}$$

Global relation ($\sim$ Green's theorem in spectral space):

$$\int_{\partial\Omega} \mathrm{e}^{-i\lambda z + \frac{\beta^2}{i\lambda}\bar{z}} \left[ (u_z + i\lambda u)dz - (u_{\bar{z}} - \frac{\beta^2}{i\lambda}u)d\bar{z} \right] = 0$$

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given $u_{|\partial\Omega}$, find $\partial_n u_{|\partial\Omega}$

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Theoretical basis for a class of competitively efficient numerical schemes - analogue of boundary element methods but in spectral space (*Fulton, Fornberg, Smitheman, Iserles, Hashemzadeh,...*)

# Spectral structure of the Laplacian

Characterising the eigenvalues/eigenfunction of the Laplacian operator $=$ solving the Helmholtz equation above (even in $\mathbb{R}^2$) is an important and difficult question with many applications:

*spectral characterization of domain geometry, billiard dynamics, ergodic theory.....*

One possible strategy is based on the analysis of the global relation for the explicit asymptotic charaterization of eigenvalues and eigenfunctions of the (Dirichlet) Laplacian on a given convex polygon.

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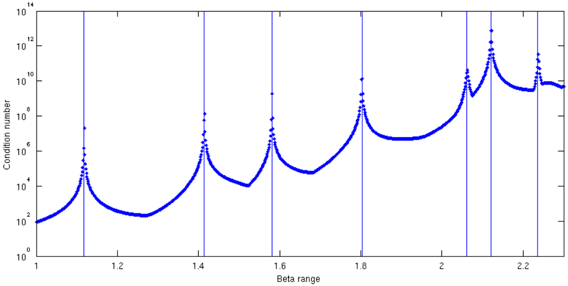
Eigenvalues and eigenfunctions have been computed explicitly *for all Robin boundary conditions* by Kalimeris and Fokas using the UT (*known since Lamé for Dirichlet conditions*)

**Recent preliminary result**: rational isosceles triangle, large eigenvalues

# Numerical evaluation of eigenvalues

A computation based on a numerical method for evaluating the Dirichlet to Neumann map from the global relation - condition number "spikes" of the matrix approximating the of the D-to-N operator correspond to eigenvalues




(*Ashton and Crooks*)

# UT inspired more general "elliptic" transforms

UT $\rightarrow$ representation of an arbitrary analytic function in a polygon:

$$u_z = \frac{1}{2\pi} \sum_{k=1}^{n} \int_{l_k} \mathrm{e}^{i\lambda z} \rho_k(\lambda) d\lambda, \quad l_k = \{\lambda : arg(\lambda) = -arg(z_k - z_{k+1})\},$$

$$\rho_k(\lambda) = \int_{z_{k+1}}^{z_k} \mathrm{e}^{-i\lambda z} u_z(z) dz, \ \ (k = 1, .., n), \qquad \sum_k \rho_k(\lambda) = 0.$$

can be extended to more general domains (*Crowdy*)

- polycircular domains, also non-convex
- domains with a mixture of straight and circular edges
- multiply connected circular domains

$\rightarrow$ important applications to fluid dynamics problem
(bubble mattresses, biharmonic problems..)

# Boundary value problems for nonlinear integrable PDEs

Using the fact that integrable PDE have a Lax pair formulation, as in the linear case one obtains

- integral representation (characterized implicitly by a linear integral equation)
- global relation

The global relation can be solved as in the linear case for a large class of boundary conditions, called linearisable

A long list of contributors:
*Fokas, Its (NLS), Shepelski, Kotlyarov, Boutet de Monvel (periodic problems), Lenells (general bc and periodic problems), P (elliptic sine-Gordon), Biondini (solitons).......*

# Bibliography

**Unified Transform Gateway** (maintained by David A Smith):
http://unifiedmethod.azurewebsites.net/