Centre for Research in Computational and Applied Mechanics

Discontinuous Galerkin Methods: An Overview and Some Applications

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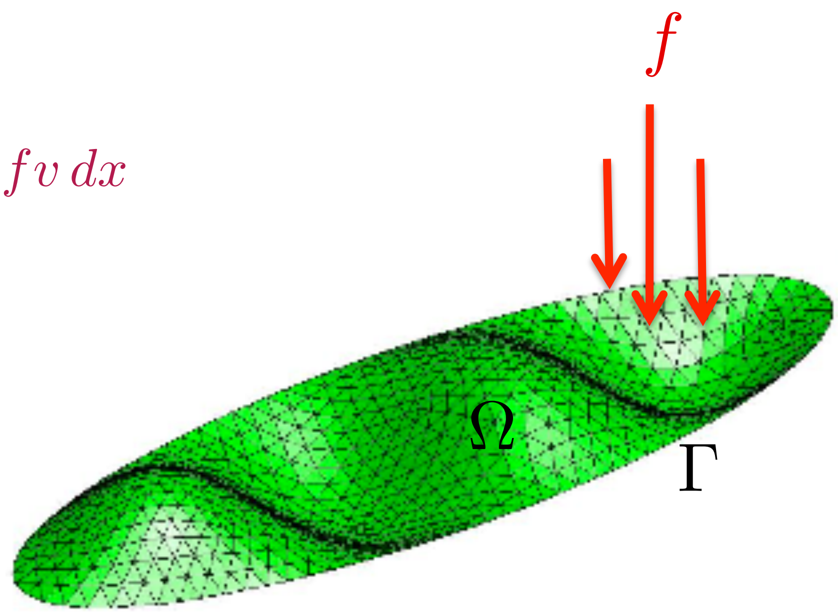
Structure of talk

- A model elliptic problem: weak or variational formulations
- The Galerkin finite element method: analysis and approximations
- Discontinuous Galerkin (DG) formulations
- Near-incompressibility in elasticity

Model problem: deformation of a membrane

Minimization of an "energy"

$$\min_v J(v) \qquad J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f v\,dx$$



The solution $\boldsymbol{u}$ satisfies the weak problem

$$\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f v\,dx$$

for all functions $\boldsymbol{v}$ which satisfy $\boldsymbol{v} = \boldsymbol{0}$ on the boundary

Sufficiently smooth u satisfies the Poisson equation and boundary condition

$$\begin{aligned} -\Delta u &= f \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

The membrane problem, continued

$$\min_{v\in V} J(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 \; dx - \int_\Omega fv \; dx$$

$$\int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega fv \; dx$$

Define the bilinear form $a(\cdot,\cdot)$ and linear functional $\ell(\cdot)$

$$a : V \times V \to \mathbb{R}\,, \qquad a(u,v) = \int_\Omega \nabla u \cdot \nabla v \; dx$$

$$\ell : V \to \mathbb{R}\,, \qquad \ell(v) = \int_\Omega fv \; dx$$

Thus the above problem is

$$\min_{v\in V} \tfrac{1}{2}a(v,v) - \ell(v)$$

or equivalently

$$a(u,v) = \ell(v) \qquad \forall v \in V$$

Interlude: the Sobolev spaces $H^m(\Omega)$

Built from the Lebesgue space of square-integrable functions:

$$L^2(\Omega) = \left\{ v \ : \ \int_\Omega v^2 \, dx := \|v\|_0^2 < \infty \right\}$$

Define, for integer $m \geq 0$,

$$H^m(\Omega) = \{ v \ : \ D^\alpha v \in L^2(\Omega) \, , \ |\alpha| \leq m \}$$

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$$

$$|\alpha| = \alpha_1 + \alpha_2$$

for problem in $\mathbb{R}^2$

Seminorm $$|v|_m^2 = \sum_{\alpha = m} \int_\Omega |D^\alpha v|^2 \, dx$$

Hilbert space with induced norm $$\|v\|_m^2 = \sum_{|\alpha| \leq m} |v|_m^2$$

$$\text{e.g.} \ \ \|v\|_1^2 = \int_\Omega \left[|v|^2 + \left(\frac{\partial v}{\partial x_1} \right)^2 + \left(\frac{\partial v}{\partial x_2} \right)^2 \right] dx$$

We will also need $H_0^1(\Omega) = \{ v \in H^1(\Omega) \ : \ v = 0 \text{ on } \Gamma \}$

Well-posedness of the variational problem

$$\min_{w \in W} \frac{1}{2} a(w,w) - \ell(w)$$

This problem has a unique solution if:

- W is a closed subspace of a Hilbert space H
- a is coercive or W-elliptic: $a(w,w) \geq \alpha \|w\|_H^2$
- a is continuous: $|a(w,z)| \leq M \|w\|_H \|z\|_H$
- ℓ is continuous: $|\ell(z)| \leq C \|z\|_H$

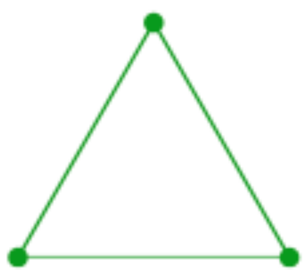
The model problem has a unique solution in $H_0^1(\Omega)$

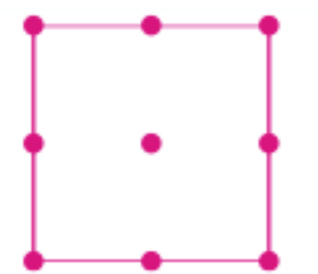
Finite element approximations

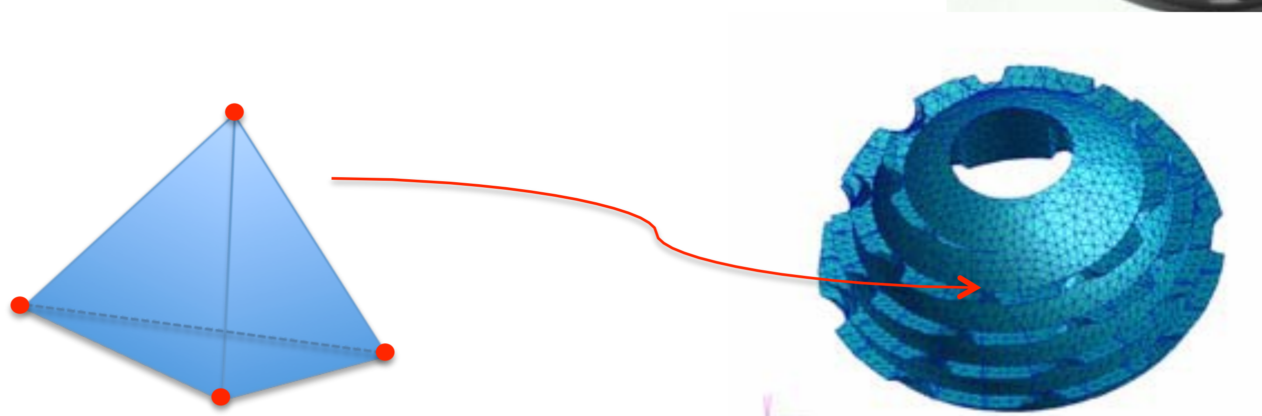
Aim: to pose the variational problem on a finite-dimensional subspace $V^h \subset V$

1. Partition the domain into subdomains or finite elements

2. Construct a basis $\{\varphi_i\}_{i=1}^{N}$ for V^h comprising continuous functions that are polynomials on each element







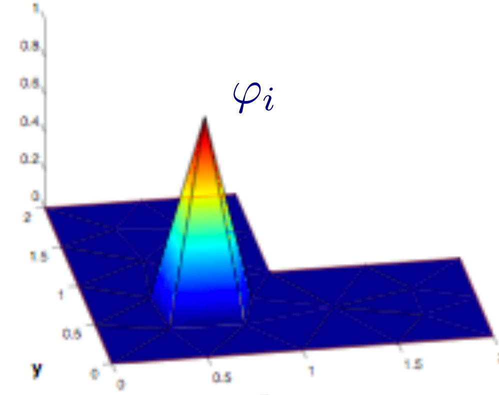
portion of hip replacement:
physical object and finite element model

The Galerkin finite element method

3. The piecewise-polynomial approximations can be written

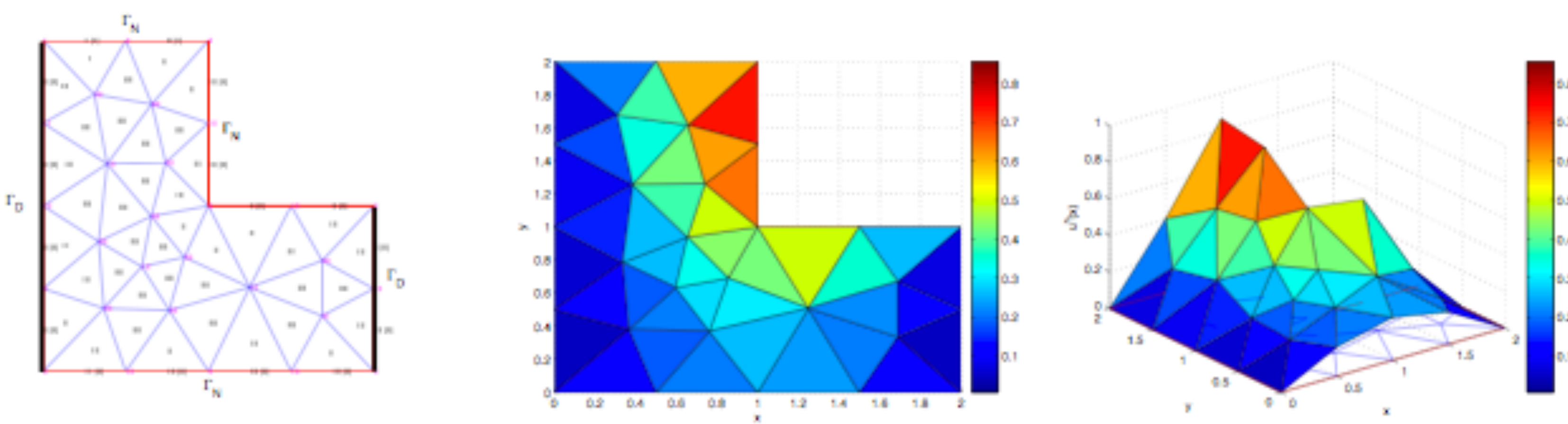
$$u_h = \sum_i \varphi_i(\boldsymbol{x}) u_i \equiv \boldsymbol{\varphi}\mathsf{u}$$

$$v_h = \sum_i \varphi_i(\boldsymbol{x}) v_i \equiv \boldsymbol{\varphi}\mathsf{v}$$



4. Substitute in the weak formulation $a(u_h, v_h) = \ell(v_h)$ to obtain

$$\sum_{i,j} v_i [a(\varphi_i, \varphi_j) u_i - \ell(\varphi_j)] = 0 \qquad \sum_i \underbrace{a(\varphi_i, \varphi_j)}_{K_{ji}} u_i = \underbrace{\ell(\varphi_j)}_{F_j} \qquad \mathsf{Ku} = \mathsf{F}$$



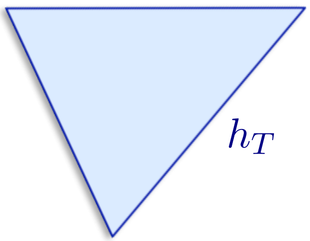
(a) Finite element mesh (b) Finite element solution (c) Finite element solution

Convergence of finite element approximations

- Construct $V_h \subset V$ and seek $u_h \in V_h$ such that for all $v_h \in V_h$

$$a(u_h, v_h) = \ell(v_h) \qquad \text{for all } v_h \in V_h \quad \Longrightarrow \quad \mathsf{Ku} = \mathsf{F}$$

- $h_T =$ diameter of T mesh size $h = \max_{T \in \mathcal{T}} h_T$



- Define the *error* by $u - u_h$: under what conditions do we have convergence in the sense that

$$\lim_{h \to 0} u_h = u?$$

- Orthogonality of the error:

$$\begin{aligned} a(u - u_h, v_h) &= a(u, v_h) - a(u_h, v_h) \\ &= \ell(v_h) - \ell(v_h) \\ &= 0 \end{aligned}$$

An a priori estimate

$$\begin{aligned}
\alpha\|u-u_h\|_V^2 &\leq a(u-u_h,u-u_h) && (V\text{-ellipticity})\\
&= a(u-u_h,u-v_h)+a(u-u_h,v_h-u_h)\\
&\leq a(u-u_h,u-v_h) && (\text{orthogonality of error})\\
&\leq M\|u-u_h\|_V\,\|u-v_h\|_V && (\text{continuity})
\end{aligned}$$

$$\|u-u_h\|_1 \leq \bar{C} \inf_{v_h\in V_h} \|u-v_h\|_1$$

Céa's lemma

Strategy for obtaining error bound: a) choose v_h to be the interpolate of u in V_h

b) use interpolation error estimate to bound actual error

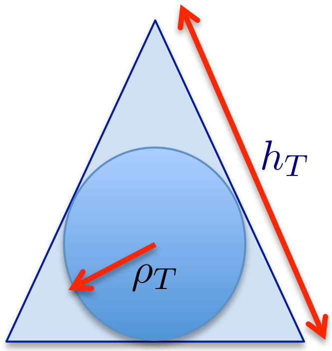
Finite element interpolation theory

Ciarlet and Raviart *Arch. Rat. Mech. Anal.* 1972

h_T = diameter of T

$\rho_T = \sup\{$diameter of B; B a ball contained in $T\}$

$\sigma_T = h_T/\rho_T$



Let $\mathcal{T}$ be a triangulation of a bounded domain Ω with polygonal boundary:

$$\bar{\Omega} = \cup_{T \in \mathcal{T}} T$$

Define the mesh size $h = \max_{T \in \mathcal{T}} h_T$

A family of triangulations is regular as $h \to 0$ if there exists $\sigma > 0$ such that

$$\sigma_T \leq \sigma \qquad \text{for all } T \in \mathcal{T}_h$$

Finite element interpolation theory

(Local estimate) For a regular triangulation with $v \in H^{k+1}(T)\,, \quad k+1 \geq m$
and the interpolation operator π which maps functions to polynomials of degree $\leq k\,,$

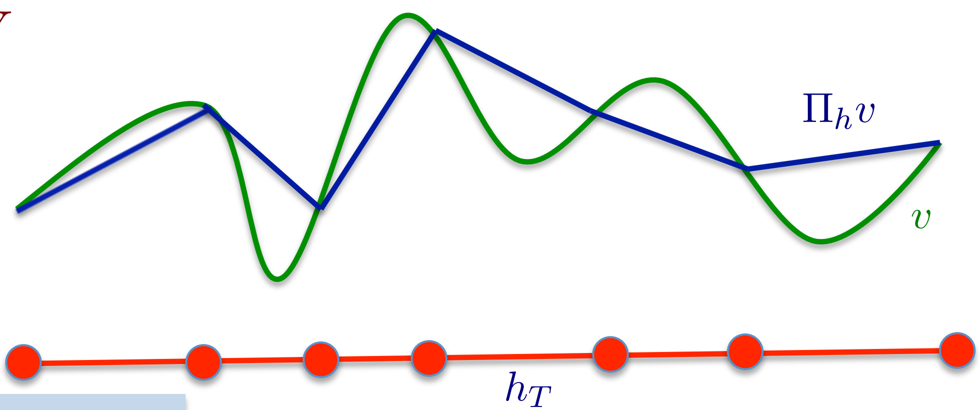
$$|v - \pi v|_{m,T} \leq C h^{k+1-m} |v|_{k+1,T}$$

(Global estimate) Let $\mathcal{T}$ be a uniformly regular triangulation of a polygonal domain. Define

$$V_h = \{v_h \in C(\bar{\Omega}) \; : \; v_h|_T \in P_k\} \cap V$$

$\Pi_h : H^2(\Omega) \to V_h$ (global interpolator)

$$h = \max_T h_T$$

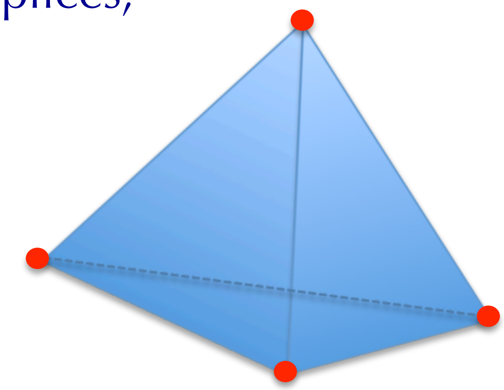


$$\|v - \Pi_h v\|_{m,\Omega} \leq C h^{k+1-m} |v|_{k+1,\Omega} \qquad m = 0, 1$$

Convergence of finite element approximations

$$\begin{aligned}\|u-u_h\|_V &\leq C\ \inf_{v_h\in V_h}\|u-v_h\|_V \\ &\leq C\|u-\Pi_h u\|_V \\ &\leq Ch^{\min(k,r-1)}|u|_r\end{aligned}$$

So for the simplest approximation, by piecewise-linear simplices,



$$\|u-u_h\|_1 \leq Ch|u|_2$$

for the second-order elliptic equations

Could use piecewise-quadratic simplices $(k=2)$ in which case

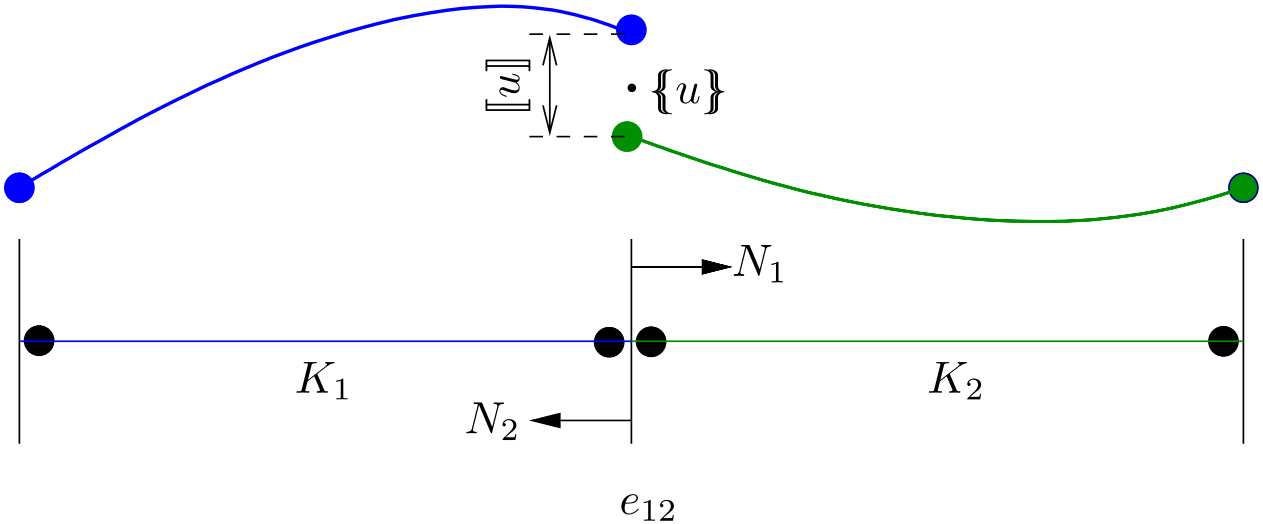
$$\|u-u_h\|_1 = O(h^2)$$

-- provided that the solution is smooth enough to belong to $H^3(\Omega)$!

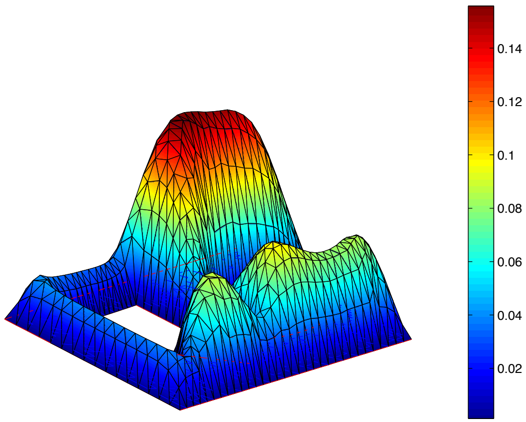
Discontinuous Galerkin (DG) methods

Discontinuous Galerkin (DG) methods

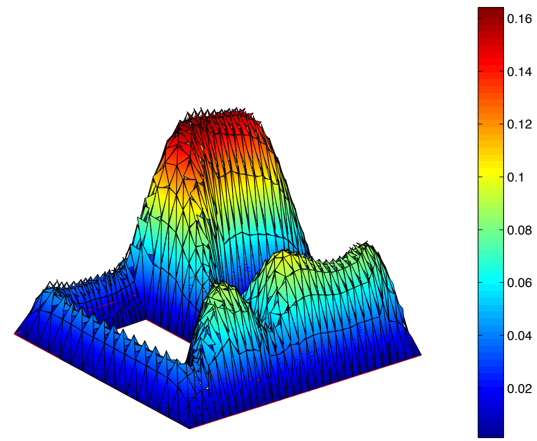
- Drop the condition of continuity
- So $V_h \notin V$
- A member $\boldsymbol{v}_h \in V_h$ is now a polynomial on each element, but not continuous across element boundaries
- An immediate consequence: much greater number of degrees of freedom



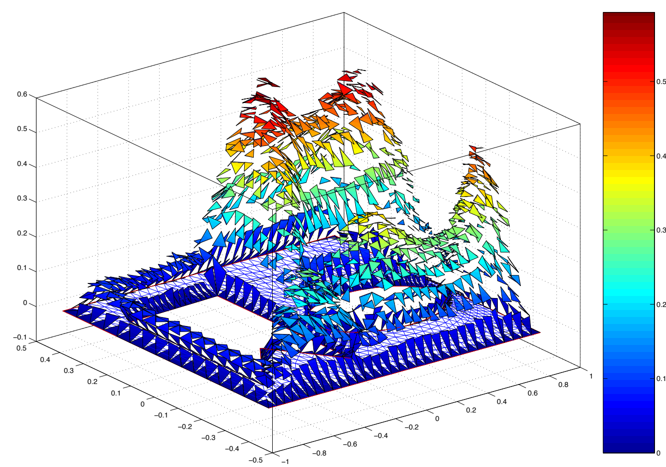
What do solutions look like?



(a) Conforming approximation



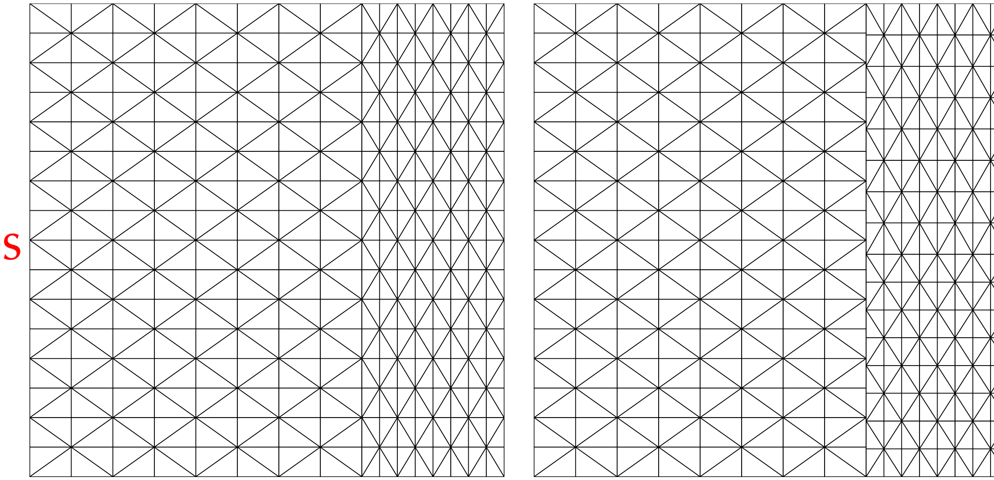
(b) Discontinuous Galerkin approximation



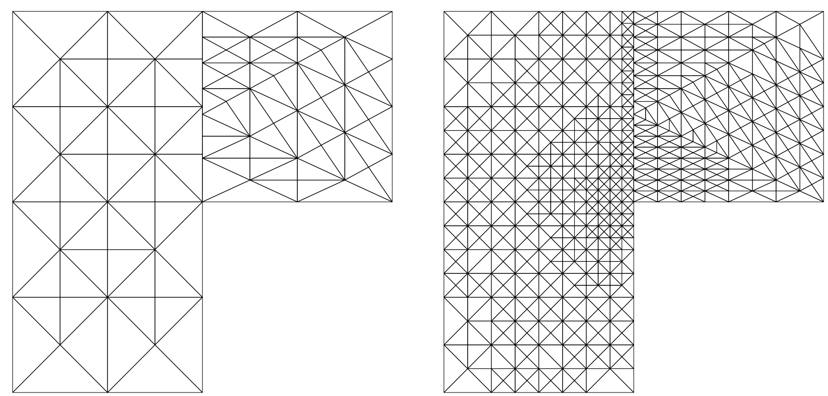
(c) Discontinuous Galerkin approximation with a

Why bother with DG?

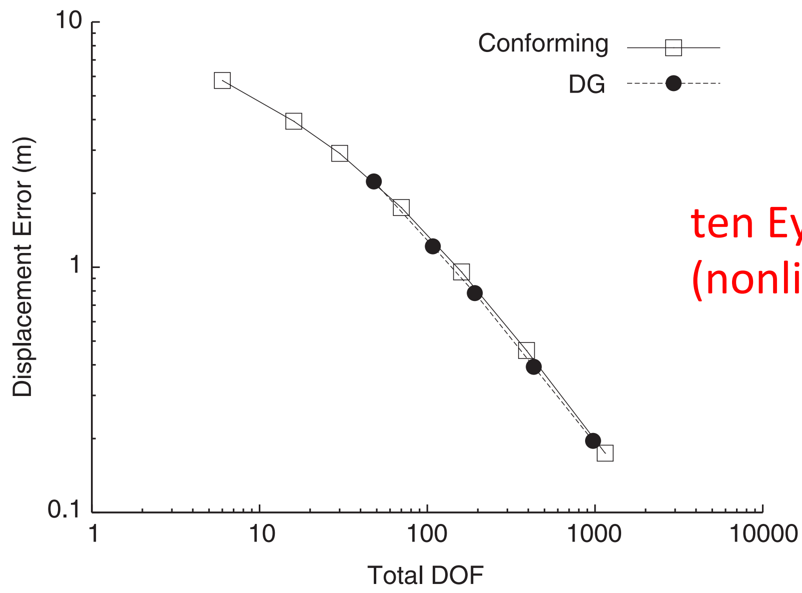
Can accommodate hanging nodes



Useful in adaptive mesh refinement



Efficiency and accuracy



ten Eyck and Lew 2010
(nonlinear elasticity)

Figure 6. Plot of the L^2-norm over $\mathscr{B}_0$ of the error in the displacement field as a function of the total number of degrees of freedom for the test in Section 6.1. Curves are shown for the discontinuous Galerkin method as well as for the conforming one. Remarkably, the two curves overlap, indicating that both methods provide the same accuracy for the same computational cost, i.e. for this example they are equally efficient.

Our objective

- For the problem of deformations of elastic bodies DG methods show good behaviour for near-incompressibility with the use of low-order triangles in two dimensions, and tetrahedra in three

- DG with quadrilaterals and hexahedra less straightforward:
  - poor behaviour, including locking, for low-order approximations

- We show why this is so, and propose some remedies

First, review derivation of heat equation

$\boldsymbol{q} =$ heat flux vector

Heat equation: Balance of energy $\operatorname{div} \boldsymbol{q} = s$

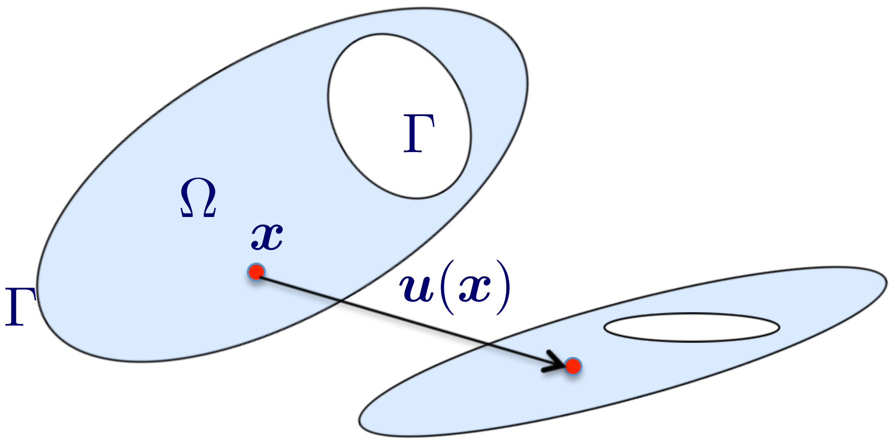
$s =$ heat source

+

Fourier's law $\boldsymbol{q} = -k \nabla \vartheta$

give the (steady) heat equation $-k \, \Delta \vartheta = s$

Governing equations for elasticity



displacement vector $\boldsymbol{u}$

stress tensor or matrix $\boldsymbol{\sigma}$

Elasticity: Equilibrium $-\mathrm{div}\,\boldsymbol{\sigma} = \boldsymbol{f}$

+

Hooke's law $\boldsymbol{\sigma} = \lambda\,\mathrm{div}\,\boldsymbol{u} + (\nabla\boldsymbol{u} + [\nabla\boldsymbol{u}]^T)$

give Navier's equation $-\bar{\Delta}\boldsymbol{u} = -[\Delta\boldsymbol{u} + (1+\lambda)\nabla\mathrm{div}\,\boldsymbol{u}] = \boldsymbol{f}$

Equivalent to minimization problem

$$\min_{\boldsymbol{u}} J(\boldsymbol{u}) = \tfrac{1}{2}\int_\Omega |\nabla_s \boldsymbol{u}|^2 + (1+\lambda)(\operatorname{div}\boldsymbol{u})^2 \, dV - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\, dV$$

$$\nabla_s \boldsymbol{u} = \tfrac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]$$

$$(\nabla_s \boldsymbol{u})_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Define $a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega [\nabla_s \boldsymbol{u} : \nabla_s \boldsymbol{v} + (1+\lambda)\operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{v}] \, dx$

$$\ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, dx$$

Then the minimization problem is $\min_{v\in V} \tfrac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) - \ell(\boldsymbol{v})$

or equivalently $a(\boldsymbol{u},\boldsymbol{v}) = \ell(\boldsymbol{v})$

which has a unique solution in $V := [H_0^1(\Omega)]^d$

Furthermore (Brenner and Sung 1992) the solution satisfies

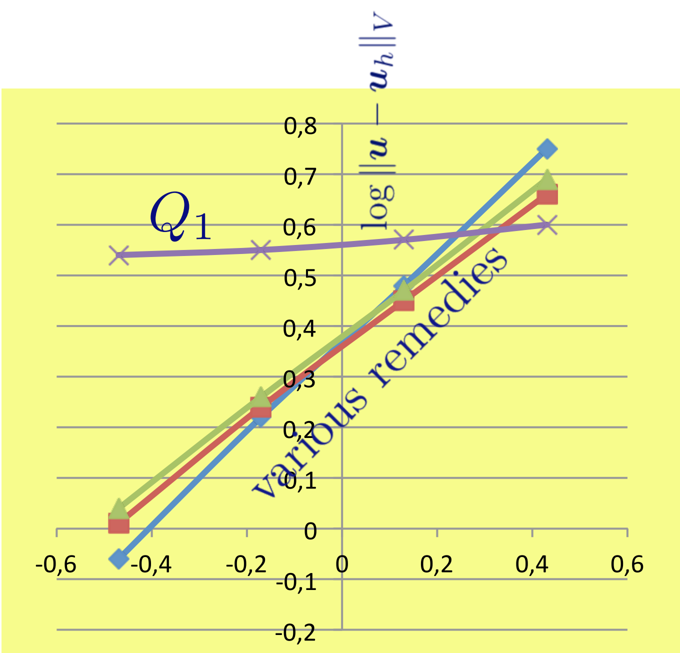
$$\boldsymbol{u} \in [H^2(\Omega)]^2 \qquad \|\boldsymbol{u}\|_{H^2} + \lambda\|\operatorname{div}\boldsymbol{u}\|_{H^1} \le C\|\boldsymbol{f}\|_{L^2}$$

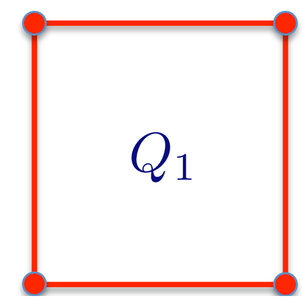
Locking

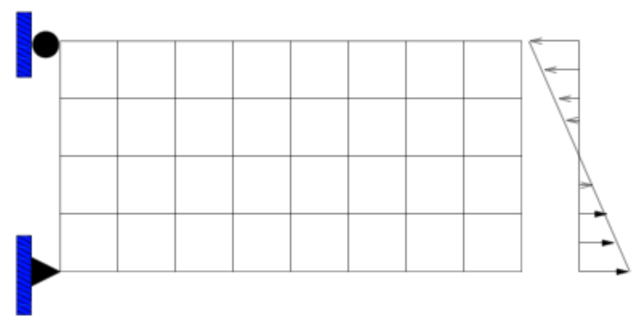
The use of low-order elements leads to a non-physical solution in the incompressible limit

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \left[\nabla_s \boldsymbol{u} : \nabla_s \boldsymbol{v} + (1+\lambda)\operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{v}\right]\, dx$$

$$\lambda \to \infty \qquad \operatorname{div}\boldsymbol{u} \to 0$$



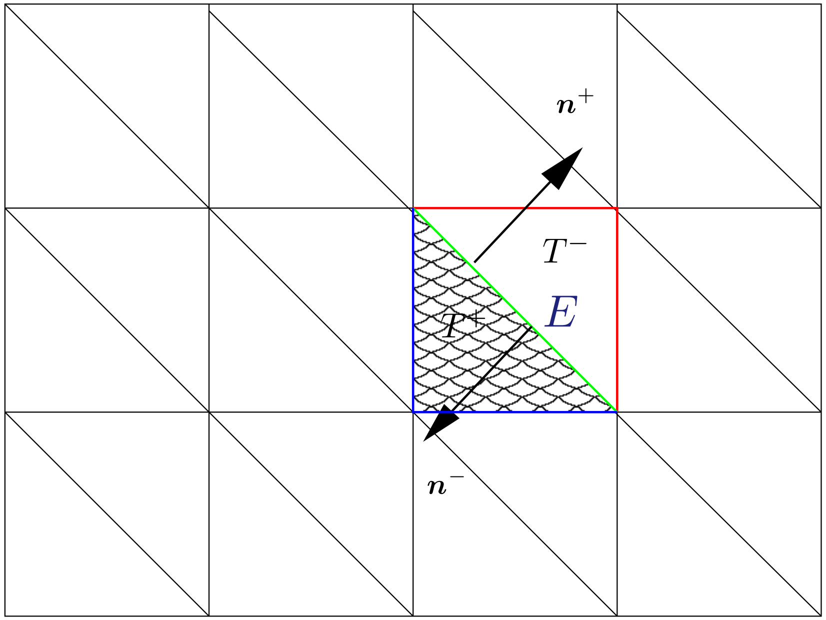




We explore the use of DG methods as a remedy for locking

DG formulation: some definitions

Jumps and averages:
on an interior edge E of element T,



$$[\boldsymbol{v}] = \boldsymbol{v}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{v}^- \cdot \boldsymbol{n}^-,$$

$$[\![\boldsymbol{v}]\!] = \boldsymbol{v}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{v}^- \otimes \boldsymbol{n}^-, \qquad \{\boldsymbol{v}\} = \tfrac{1}{2}(\boldsymbol{v}^+ + \boldsymbol{v}^-)$$

$$[\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+\boldsymbol{n}^+ + \boldsymbol{\tau}^-\boldsymbol{n}^-, \qquad \{\boldsymbol{\tau}\} = \tfrac{1}{2}(\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-)$$

For example $[\![\boldsymbol{\tau}]\!] = (\boldsymbol{\tau}^+ - \boldsymbol{\tau}^-)\boldsymbol{n}^+$

On an edge that forms part of the boundary, $[\![\boldsymbol{v}]\!] = \boldsymbol{v} \otimes \boldsymbol{n}$

$$\{\boldsymbol{\tau}\} = \boldsymbol{\tau}\boldsymbol{n}$$

Setting it up

$$-\bar{\Delta}\boldsymbol{u} = \underbrace{-\left[\Delta\boldsymbol{u} + (1+\lambda)\nabla \mathrm{div}\,\boldsymbol{u}\right]}_{-\mathrm{div}\,\sigma} = \boldsymbol{f}$$

$$\boldsymbol{\sigma} = \lambda\,\mathrm{div}\,\boldsymbol{u} + (\nabla\boldsymbol{u} + [\nabla\boldsymbol{u}]^T)$$

- take the inner product with a test function
- integrate, and integrate by parts

$$-\int_\Omega \mathrm{div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v}\;dV = \int_V \boldsymbol{f}\cdot\boldsymbol{v}\;dV$$

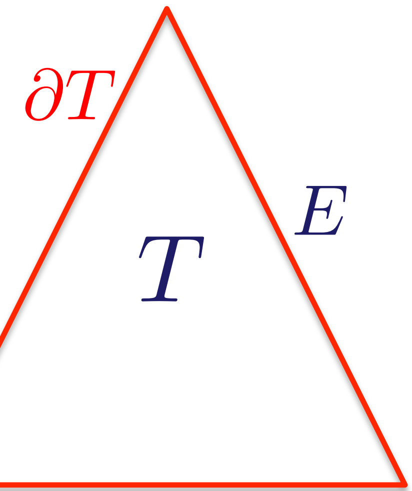
Consider the left-hand side:

$$\int_\Omega \mathrm{div}\;\boldsymbol{\sigma}\cdot\boldsymbol{v}\;dV = \sum_T\int_T \mathrm{div}\;\boldsymbol{\sigma}\cdot\boldsymbol{v}\;dV$$

$$\int_\Omega \mathrm{div}\;\boldsymbol{\sigma}\cdot\boldsymbol{v}\;dV = \sum_T\int_{\partial T}\boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{v}\;ds - \sum_T\int_T\boldsymbol{\sigma}:\nabla_s\boldsymbol{v}\;dV$$

Thus weak equation is now

$$-\sum_T \int_{\partial T} \boldsymbol{\sigma n} \cdot \boldsymbol{v} \; ds + \sum_T \int_T \boldsymbol{\sigma} : \nabla_s \boldsymbol{v} \; dV = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \; dV$$



Next, we need the "magic formula"

$$\sum_T \int_{\partial T} \boldsymbol{v} \cdot \boldsymbol{\sigma n} \; ds = \sum_E \int_E [\![\boldsymbol{v}]\!] : \{\boldsymbol{\sigma}\} \; ds + \sum_{E_{\text{int}}} \int_{E_{\text{int}}} \{\boldsymbol{v}\} \cdot [\![\boldsymbol{\sigma}]\!] \; ds$$

This gives

$$-\sum_{E}\int_{E}[\![\boldsymbol{v}]\!]:\{\boldsymbol{\sigma}\}\;ds-\sum_{E_{\mathrm{int}}}\int_{E_{\mathrm{int}}}\{\boldsymbol{v}\}\cdot[\![\boldsymbol{\sigma}]\!]\;ds+\int_{\Omega}\boldsymbol{\sigma}:\nabla_{s}\boldsymbol{v}\;dV=\int_{\Omega}\boldsymbol{f}\cdot\boldsymbol{v}\;dV$$

Since the exact solution is smooth ($\boldsymbol{u}\in[H^2(\Omega)]^d$) and the stress is continuous we can assume that

$$[\![\boldsymbol{\sigma}]\!]=\mathbf{0}$$

which leaves

$$-\sum_{E}\int_{E}[\![\boldsymbol{v}]\!]:\{\boldsymbol{\sigma}\}\;ds+\int_{\Omega}\boldsymbol{\sigma}:\nabla_{s}\boldsymbol{v}\;dV=\int_{\Omega}\boldsymbol{f}\cdot\boldsymbol{v}\;dV$$

Exploiting again the smoothness of the exact solution we can add a term to symmetrize the problem:

$[\![\boldsymbol{u}]\!] = \mathbf{0}$

$$-\sum_E \int_E [\![\boldsymbol{v}]\!] : \{\boldsymbol{\sigma}(\boldsymbol{u})\}\, ds - \sum_E \int_E [\![\boldsymbol{u}]\!] : \{\boldsymbol{\sigma}(\boldsymbol{v})\}\, ds$$

$$+ \sum_E \int_\Omega \boldsymbol{\sigma} : \nabla_s \boldsymbol{v}\, dV = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dV$$

Finally, we may need to stabilize the problem: this gives us the **symmetric interior penalty (SIP)** formulation (Douglas and Dupont 1976)

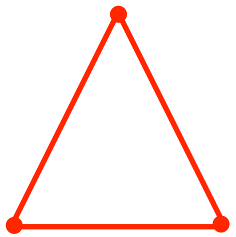
$$-\sum_E \int_E [\![\boldsymbol{v}]\!] : \{\boldsymbol{\sigma}(\boldsymbol{u})\}\, ds - \sum_E \int_E [\![\boldsymbol{u}]\!] : \{\boldsymbol{\sigma}(\boldsymbol{v})\}\, ds + k \sum_E \int_E \frac{1}{h_E} [\![\boldsymbol{u}]\!] [\![\boldsymbol{v}]\!]$$

$$+ \sum_E \int_\Omega \boldsymbol{\sigma} : \nabla_s \boldsymbol{v}\, dV = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dV$$

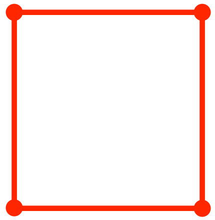
The DG formulation

$$V_h = \left\{ \boldsymbol{v}_h \ : \ \boldsymbol{v}_h \in [L^2(\Omega)]^d \, , \boldsymbol{v}_h|_T \in P_1(T) \text{ or } Q_1(T) \right\}$$

$$P_1(T): \quad \boldsymbol{v}_h = \boldsymbol{a}_0 + \boldsymbol{a}_1 x + \boldsymbol{a}_2 y$$



$$Q_1(T): \quad \boldsymbol{v}_h = \boldsymbol{a}_0 + \boldsymbol{a}_1 x + \boldsymbol{a}_2 y + \boldsymbol{a}_3 xy$$



$$V_h = \{\boldsymbol{v}_h \ : \ \boldsymbol{v}_h \in [L^2(\Omega)]^d, \boldsymbol{v}_h|_T \in P_1(T) \text{ or } Q_1(T)\}$$

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h)$$

$$a_h(\boldsymbol{u}, \boldsymbol{v})$$

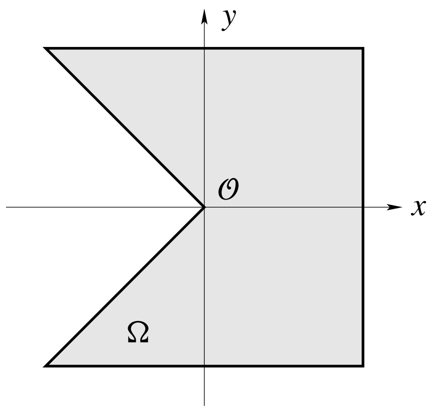
$$\sum_T \int_T \boldsymbol{\sigma}(\boldsymbol{u}) : \nabla_s \boldsymbol{v} \; dV + \theta \sum_E \int_E [\![\boldsymbol{u}]\!] : \{\boldsymbol{\sigma}(\boldsymbol{v})\} \, ds - \sum_E \int_E [\![\boldsymbol{v}]\!] : \{\boldsymbol{\sigma}(\boldsymbol{u})\} \, ds + k \sum_E \int_E \frac{1}{h_E} [\![\boldsymbol{u}]\!][\![\boldsymbol{v}]\!] = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \; dV$$

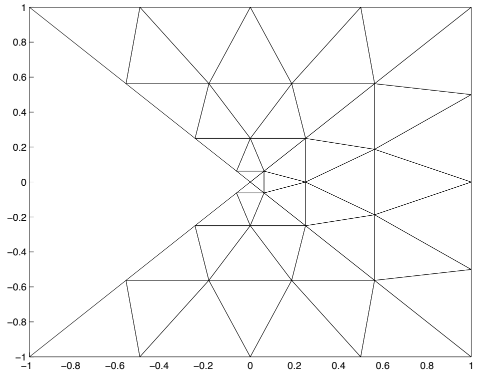
$\theta = \begin{cases} +1 \\ -1 \\ 0 \end{cases}$

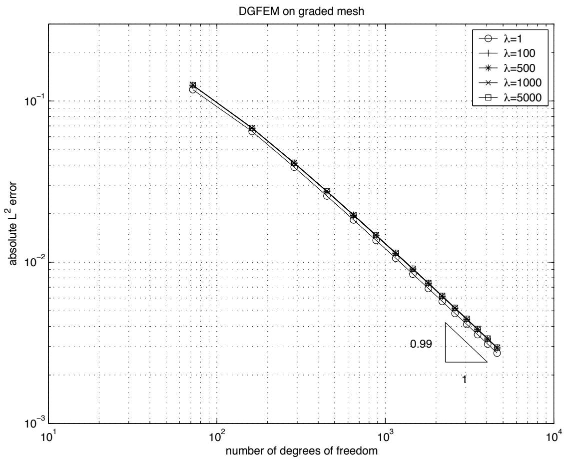
- $+1$ Nonsymmetric Interior Penalty Galerkin (NIPG)
  Rivière & Wheeler 1999, 2000; Oden, Babuska & Baumann 1998
- -1 Symmetric Interior Penalty Galerkin (SIPG)
  Douglas & Dupont 1976, Arnold 1982, Hansbo & Larson 2002
- 0 Incomplete Interior Penalty Galerkin (IIPG)
  Dawson, Sun and Wheeler 2004

DG with triangles is uniformly convergent

WIHLER 2002







Some computational results with quads appear to show locking-free behaviour …

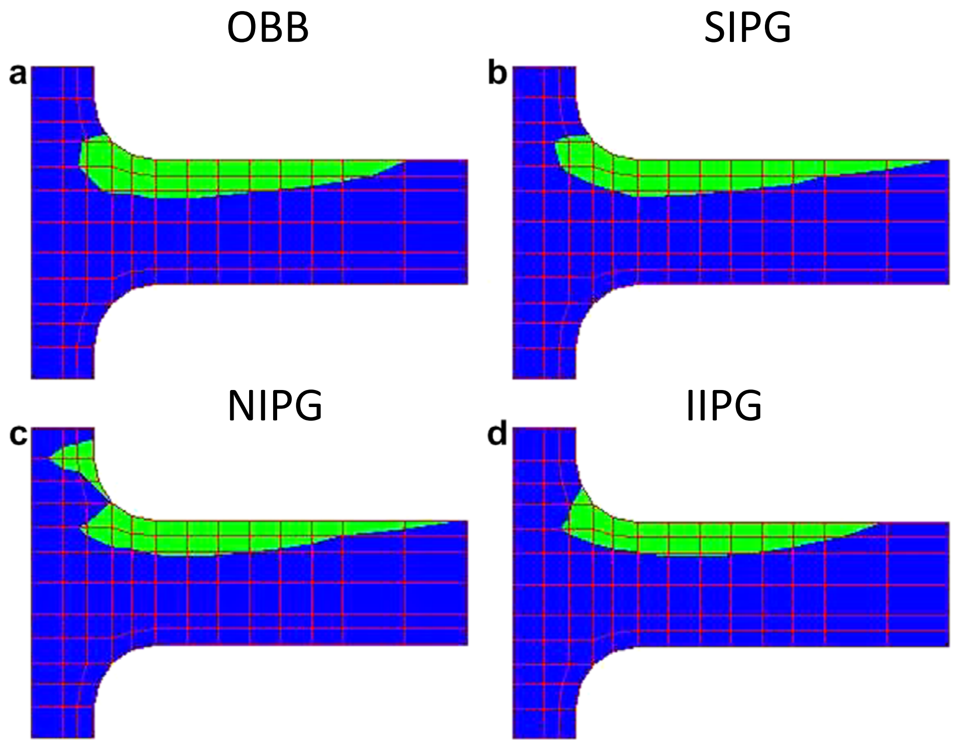
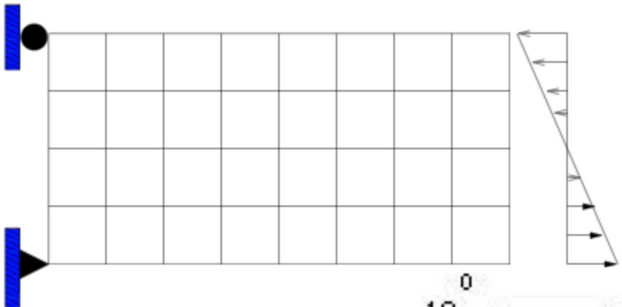


Fig. 8. Tensile stress contours ($\sigma_y > 0.15$ MPa) of DG solutions of beam problem with Poisson's ratio 0.499. (a) OBB; (b) SIPG; $\delta_p = 30$; (c) NIPG; $\delta_p = 10$; and (d) IIPG; $\delta_p = 50$.

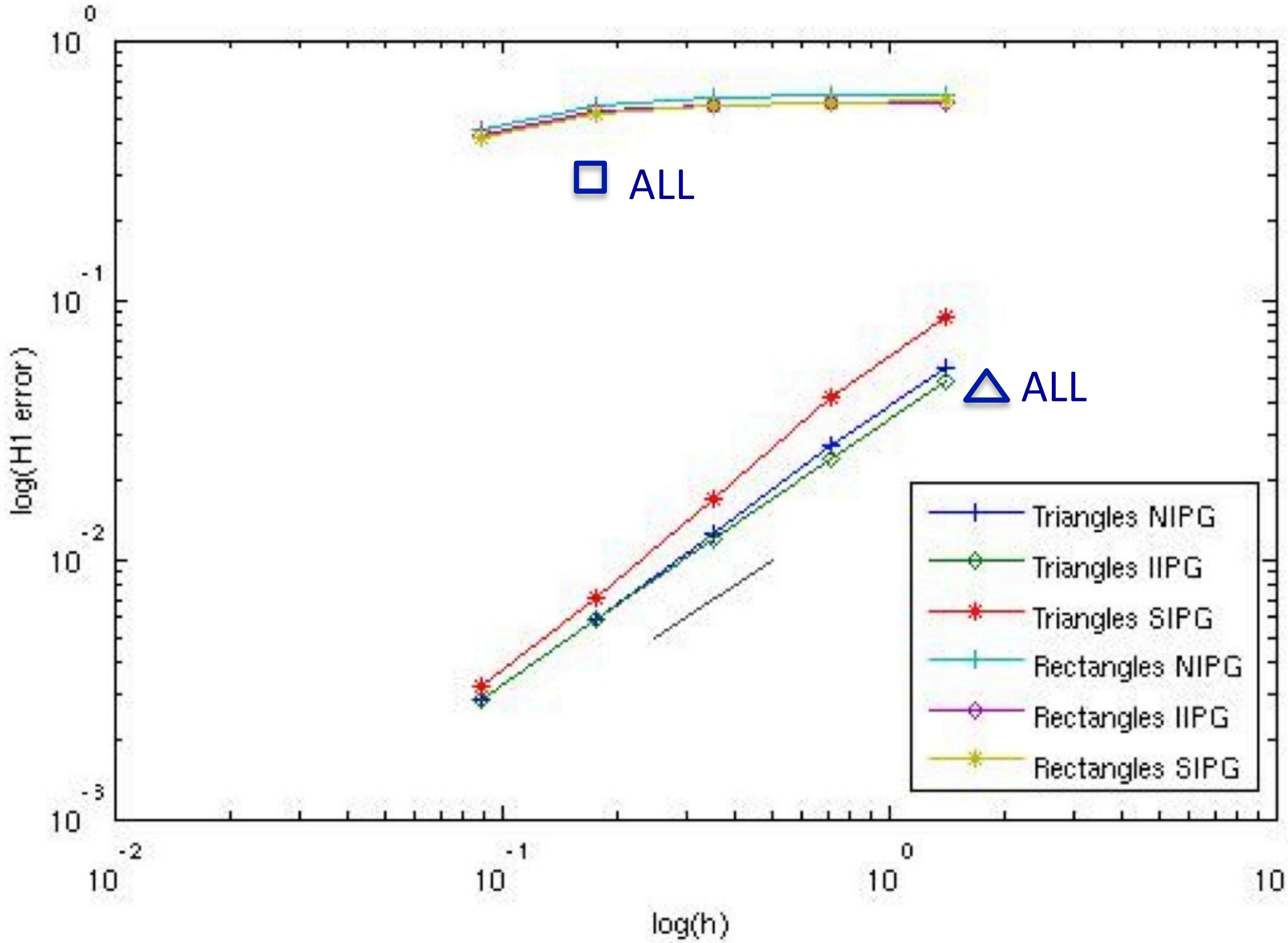
Liu, Wheeler and Dawson 2009

These authors report good results using all methods

… but not so in other cases



Grieshaber, McBride and R. (2015)



- We know that DG works for simplicial elements
- Some authors report good results for quads
- We find locking for a simple example
- What's going on?

Let's see why DG with triangles works

$$a_h(\boldsymbol{u},\boldsymbol{v}) := \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\sigma}(\boldsymbol{u}) : \nabla_s \boldsymbol{v} + \theta \sum_{E\in\Gamma}\int_E [\![\boldsymbol{u}]\!] : \{\boldsymbol{\sigma}(\boldsymbol{v})\} - \sum_{E\in\Gamma}\int_E \{\boldsymbol{\sigma}(\boldsymbol{u})\} : [\![\boldsymbol{v}]\!]$$

$$+k_\mu \mu \sum_{E\in\Gamma}\int_E \frac{1}{h_E}[\![\boldsymbol{u}]\!] : [\![\boldsymbol{v}]\!] + k_\lambda \lambda \sum_{E\in\Gamma}\frac{1}{h_E}\int_E [\![\boldsymbol{u}]\!] : [\![\boldsymbol{v}]\!]$$

(note the use of two stabilization terms)

$$:= \quad 2\mu\left[\sum_{T\in\mathcal{T}_h}\int_T \nabla_s \boldsymbol{u} : \nabla_s \boldsymbol{v}\; dx + \theta\sum_{E\in\Gamma}\int_E [\![\boldsymbol{u}]\!] : \{\nabla_s \boldsymbol{v}\} - \sum_{E\in\Gamma}\int_E \{\nabla \boldsymbol{u}\} : [\![\boldsymbol{v}]\!] + k_\mu \sum_{E\in\Gamma}\int_E \frac{1}{2h_E}[\![\boldsymbol{u}]\!] : [\![\boldsymbol{v}]\!]\right]$$

$$+\lambda\left[+\sum_{T\in\mathcal{T}_h}\int_T (\operatorname{div}\boldsymbol{u})(\operatorname{div}\boldsymbol{v}) + \theta\sum_{E\in\Gamma}\int_E [\![\boldsymbol{u}]\!] : \{(\operatorname{div}\boldsymbol{v})\boldsymbol{I}\} - \sum_{E\in\Gamma}\int_E \{(\operatorname{div}\boldsymbol{u})\boldsymbol{I}\} : [\![\boldsymbol{v}]\!]\; ds\right.$$

(I) (II) (III)

$$\left. + k_\lambda \sum_{E\in\Gamma}\frac{1}{h_E}\int_E [\![\boldsymbol{u}]\!] : [\![\boldsymbol{v}]\!]\right]$$

(IV)

Error analysis for linear triangles: NIPG

Approximation error $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$

$\Pi\boldsymbol{u}$ = interpolant of $\boldsymbol{u}$

$$= \underbrace{\boldsymbol{u} - \Pi\boldsymbol{u}}_{\boldsymbol{\eta}} \quad + \quad \underbrace{\Pi\boldsymbol{u} - \boldsymbol{u}_h}_{\boldsymbol{\xi}}$$

interpolation error

Use Crouzeix-Raviart interpolant: linear on triangles,
continuous at midpoints of edges

Properties:

$$\int_E (\boldsymbol{u} - \Pi\boldsymbol{u}) \cdot \boldsymbol{n} \; ds = 0$$

$$\int_T \mathrm{div}(\boldsymbol{u} - \Pi\boldsymbol{u}) \; dV = 0$$

$$
\begin{aligned}
\boldsymbol{e} &= \boldsymbol{\eta} + \boldsymbol{\xi} \\
\boldsymbol{\eta} &= \boldsymbol{u} - \Pi \boldsymbol{u} \\
\boldsymbol{\xi} &= \Pi \boldsymbol{u} - \boldsymbol{u}_h
\end{aligned}
$$

$$
\boldsymbol{\xi} \in V_h \quad \Rightarrow \quad \operatorname{div} \boldsymbol{\xi}, \quad \boldsymbol{\sigma}(\boldsymbol{\xi}) \quad \text{const.}
$$

λ–terms in error bound can be handled as follows: for example,

$$
\text{(I)} = \lambda \sum_T \int_T (\operatorname{div} \boldsymbol{\eta})(\operatorname{div} \boldsymbol{\xi}) \; ds = \lambda \sum_T \operatorname{div} \boldsymbol{\xi} \int_T \operatorname{div} \boldsymbol{\eta} \; ds = 0
$$

to give eventually

$$
\|\boldsymbol{e}\|_{\mathrm{DG}}^2 \quad \leq \quad Ch^2 \left(\|\boldsymbol{u}\|_{H^2(\Omega)}^2 + \lambda^2 \| \operatorname{div} \boldsymbol{u}\|_{H^1(\Omega)}^2 \right)
$$

Use estimate $\|\boldsymbol{u}\|_{H^2} + \lambda \| \operatorname{div} \boldsymbol{u}\|_{H^1} \leq C \|\boldsymbol{f}\|_{L^2}$ to get uniform convergence

Convergence analysis for quadrilaterals

First need to construct a suitable interpolant $\Pi \boldsymbol{u} \in V_h$

As before $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$

$$= \underbrace{\boldsymbol{u} - \Pi \boldsymbol{u}}_{\boldsymbol{\eta}} + \underbrace{\Pi \boldsymbol{u} - \boldsymbol{u}_h}_{\boldsymbol{\xi}}$$

and $\Pi \boldsymbol{u}$ must satisfy

$$\int_E (\boldsymbol{u} - \Pi \boldsymbol{u}) \cdot \boldsymbol{n} \; ds = 0$$

$$\int_E [\![\boldsymbol{u} - \Pi \boldsymbol{u}]\!] \cdot \boldsymbol{n} \; ds = 0$$

$$\int_{\partial T} (\boldsymbol{u} - \Pi \boldsymbol{u}) \cdot \boldsymbol{n} \; ds = 0$$

The interpolant

Inspired by DOUGLAS, SANTOS, SHEEN & YE 1999; CIA, DOUGLAS, YE 1999:

Construct interpolant with span

$$\left\{ \begin{array}{c} 1, x, y, \theta(x) + \kappa(y) \\ 1, x, y, \kappa(x) + \theta(y) \end{array} \right\}$$

$$\begin{aligned} \theta(x) &= 3x^2 - 10x^4 + 7x^6 \\ \kappa(x) &= -3x^2 + 5x^4 \end{aligned}$$

Define $\Pi \boldsymbol{u}$ as orthogonal projection on element

Error bound

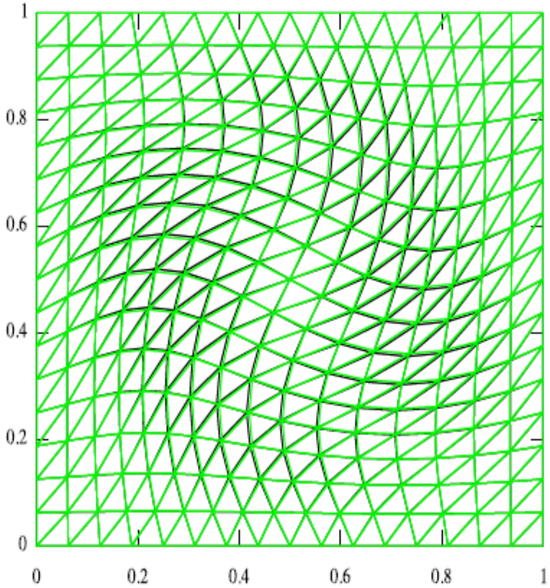
Eventually get

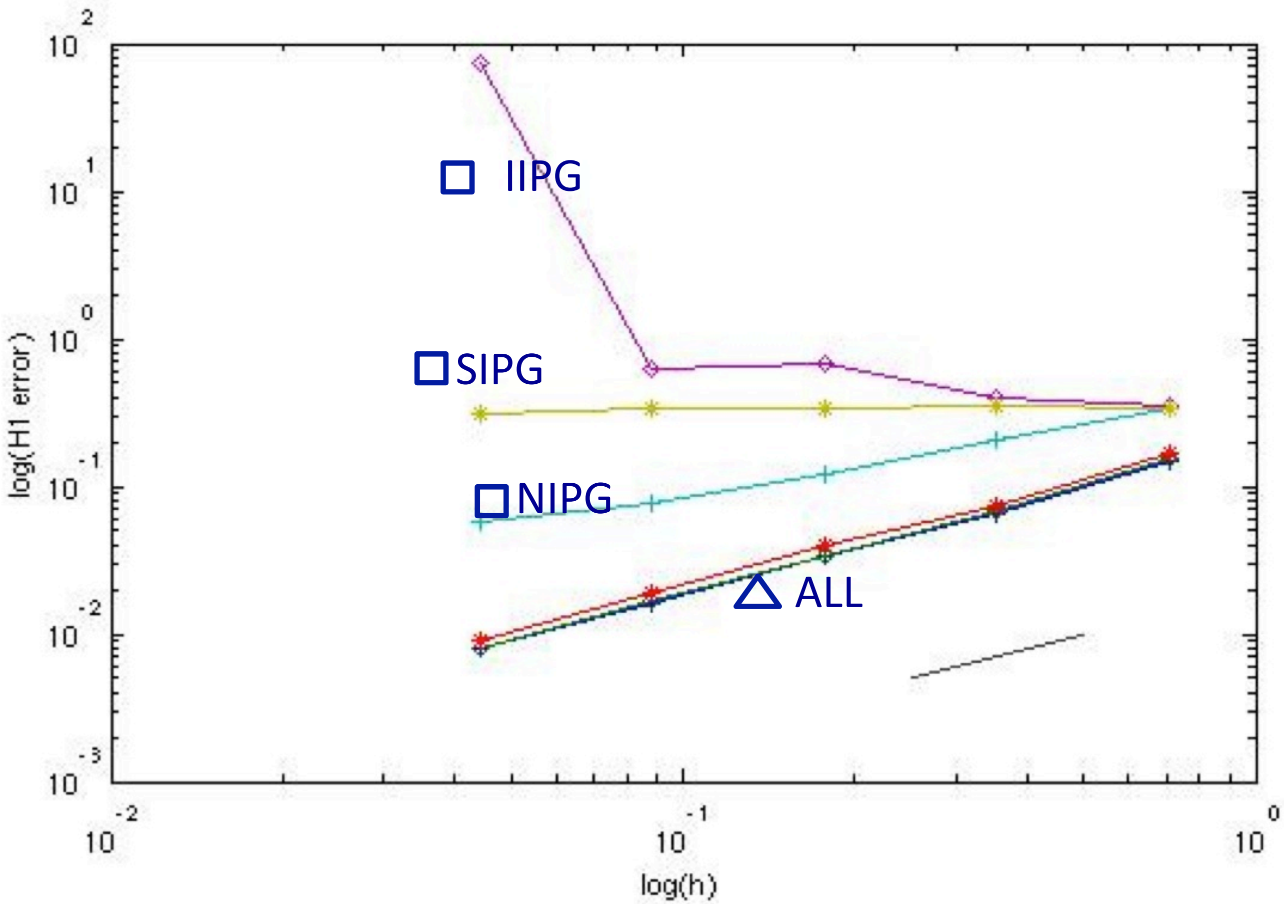
$$\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h\|_{\mathrm{DG}}^2 \leq C \sum_T h_T^2 \left(\|\boldsymbol{u}\|_{H^2(T)}^2 + \underline{\lambda^2 \|\boldsymbol{u}\|_{H^2(T)}^2} + \lambda^2 \| \operatorname{div} \boldsymbol{u}\|_{H^1(T)}^2 \right)$$

so not possible to bound λ-dependent term

Example: square plate

BRENNER *SINUM* 1993





$$\Omega = (0,1)^2 \qquad \mu = 1$$

$$\begin{aligned} u_1(x,y) &= \sin 2\pi y(-1+\cos 2\pi x) + \frac{1}{1+\lambda}\sin \pi x \sin \pi y \\ u_2(x,y) &= \sin 2\pi x(1-\cos 2\pi y) + \frac{1}{1+\lambda}\sin \pi x \sin \pi y \end{aligned}$$

A remedy: selective reduced integration (SRI)

Recall classical SRI for elasticity problem:

to overcome volumetric locking one replaces

$$\int_\Omega \left[\nabla_s \boldsymbol{u}_h : \nabla_s \boldsymbol{v}_h + \underbrace{\lambda(\operatorname{div} \boldsymbol{u})(\operatorname{div} \boldsymbol{v})}_{\text{volumetric term}}\right] \, dV$$

by

$$\int_\Omega \left[\nabla_s \boldsymbol{u}_h : \nabla_s \boldsymbol{v}_h \, dV + \underbrace{\lambda \sum_T w_T \operatorname{div} \boldsymbol{u}(\boldsymbol{x}_T) \operatorname{div} \boldsymbol{v}(\boldsymbol{x}_T)}_{\text{underintegration of volumetric term}}\right.$$

or

$$\int_\Omega \left[\nabla_s \boldsymbol{u}_h : \nabla_s \boldsymbol{v}_h \, dV + \lambda \sum_T \int_T (\Pi_0 \operatorname{div} \boldsymbol{u})(\Pi_0 \operatorname{div} \boldsymbol{v}) \, dV\right.$$

$$\Pi_0 \operatorname{div} \boldsymbol{v} = \frac{1}{|T|} \int_T \operatorname{div} \boldsymbol{v} \, dV$$

Underintegration applied to (II) gives

$$\begin{aligned} \theta\lambda \int_E [\![\boldsymbol{\xi}]\!]\{\operatorname{div}\boldsymbol{\eta}\}\, ds &\simeq \theta\lambda \int_E \Pi_0[\![\boldsymbol{\xi}]\!]\,\Pi_0\{\operatorname{div}\boldsymbol{\eta}\}\, ds \\ &= \theta\lambda\,\Pi_0\{\operatorname{div}\boldsymbol{\eta}\} \int_E \Pi_0[\![\boldsymbol{\xi}]\!]\, ds \\ &= \theta\lambda\,\Pi_0\{\operatorname{div}\boldsymbol{\eta}\} \int_E [\![\boldsymbol{\xi}]\!]\, ds = 0 \end{aligned}$$

Replace (IV) with

$$\begin{aligned} k_\lambda\lambda\theta \sum_E \int_E \Pi_0[\![\boldsymbol{\xi}]\!]\,\Pi_0[\![\boldsymbol{\eta}]\!] &= k_\lambda\lambda\theta \sum_E \int_E [\![\Pi_0\boldsymbol{\xi}]\!]\,[\![\boldsymbol{\eta}]\!] \\ &= k_\lambda\lambda\theta \sum_E \Pi_0[\![\boldsymbol{\xi}]\!] \int_E [\![\boldsymbol{\eta}]\!] \\ &= 0 \end{aligned}$$

But we now have to check **coercivity** and **consistency** of the modified bilinear form!

Underintegration applied to (II) gives

$$\begin{aligned}\theta\lambda \int_E [\![\boldsymbol{\xi}]\!]\{\mathrm{div}\,\boldsymbol{\eta}\}\ ds &\simeq \theta\lambda \int_E \Pi_0[\![\boldsymbol{\xi}]\!]\,\Pi_0\{\mathrm{div}\,\boldsymbol{\eta}\}\ ds \\ &= \theta\lambda\,\Pi_0\{\mathrm{div}\,\boldsymbol{\eta}\}\int_E \Pi_0[\![\boldsymbol{\xi}]\!]\ ds \\ &= \theta\lambda\,\Pi_0\{\mathrm{div}\,\boldsymbol{\eta}\}\int_E [\![\boldsymbol{\xi}]\!]\ ds = 0\end{aligned}$$

Eventually get

$$a_h(\boldsymbol{\eta},\boldsymbol{\xi}) \le C\|\boldsymbol{\xi}\|_{\mathrm{DG}}\Big(\textstyle\sum_T h_T^2\Big(\underbrace{\|\boldsymbol{u}\|_{H^2}^2 + \lambda^2\|\mathrm{div}\,\boldsymbol{u}\|_{H^1}^2}_{\le C\|\mathbf{f}\|^2}\Big)\Big)^{1/2}$$

But we now have to check **coercivity** and **consistency** of the modified bilinear form!

Coercivity

NIPG: OK for SRI on terms (II) and (III)

SIPG: OK for SRI on terms (II), (III), (IV)

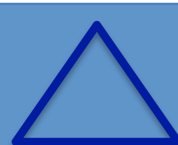
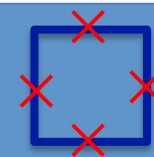
IIPG: OK for SRI on terms (III) and (IV)

Term (III) when underintegrated leads to a **consistency error**

$$|E^{\mathrm{RI}}(\boldsymbol{u},\boldsymbol{\xi})| = \lambda\Big|\sum_E \int_E \big(\Pi_0\{\mathrm{div}\,\boldsymbol{u}\}\,\mathrm{tr}\,\Pi_0[\![\boldsymbol{\xi}]\!] - \{\mathrm{div}\,\boldsymbol{u}\}\,\mathrm{tr}[\![\boldsymbol{\xi}]\!]\big)ds\Big|$$

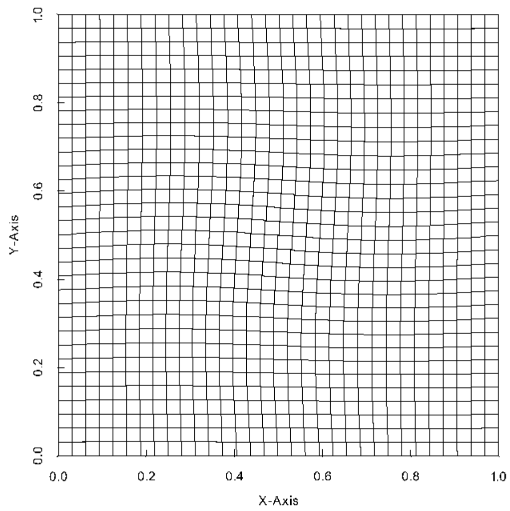
which can be controlled

Locking-free (uniformly convergent) behaviour

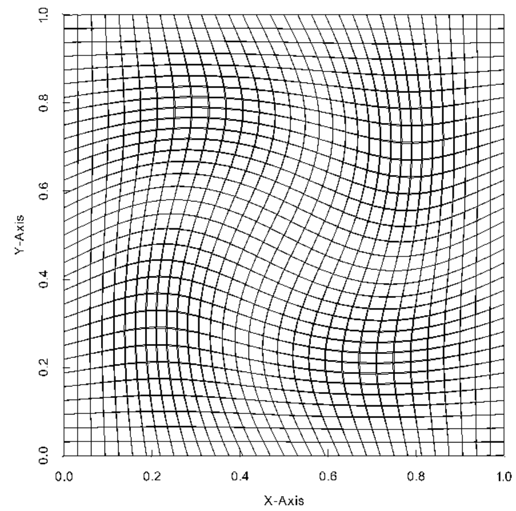
			
	$\mathcal{P}_1$	$\mathcal{Q}_1$	$\mathcal{Q}_1 + \mathrm{SRI}$
SIP	✔	✖	✔
NIP	✔	✖	✔
IIP	✔	✖	✔

Square plate:

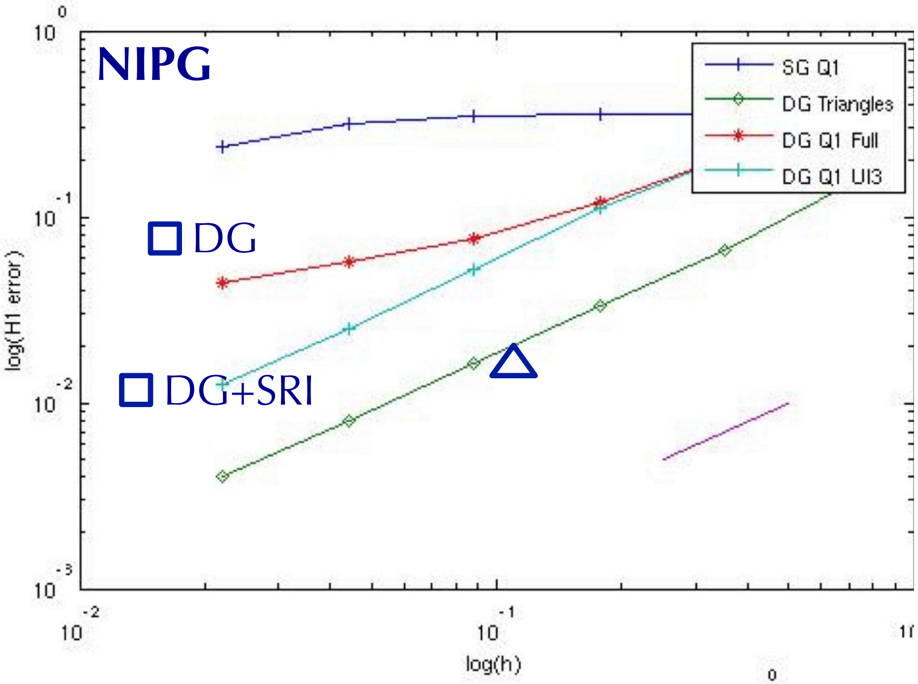
DG with triangles
quads with and without SRI

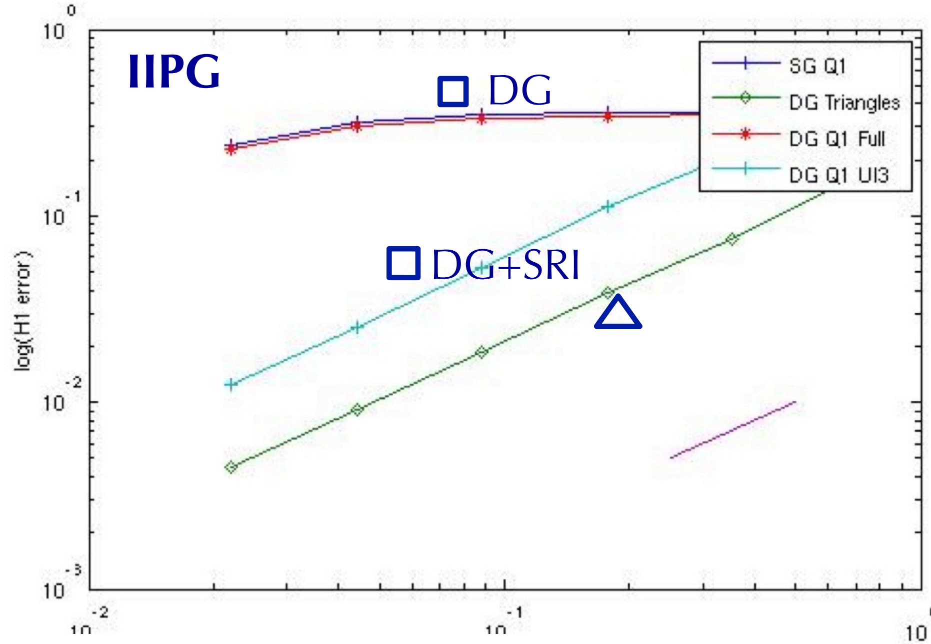


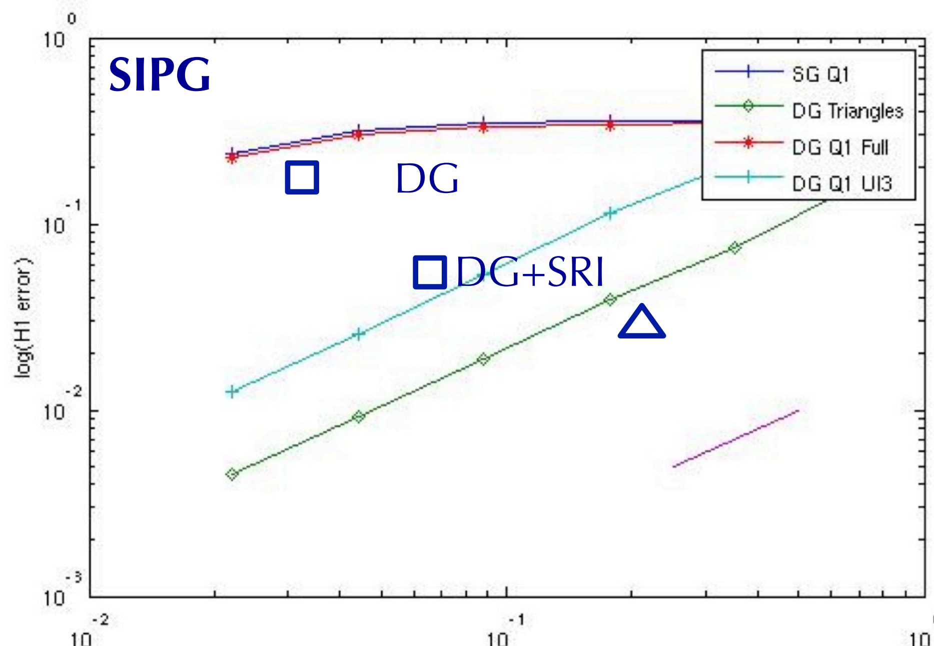
Standard finite elements /
DG quads without SRI



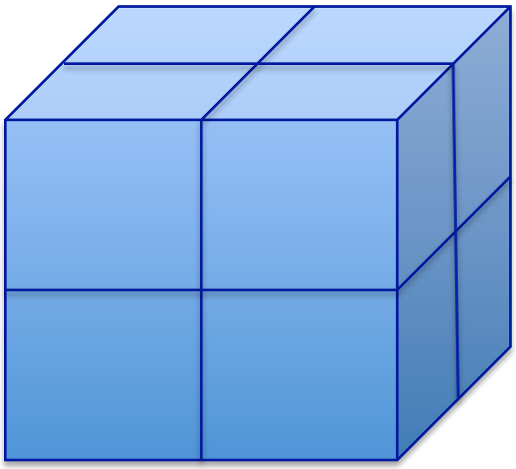
DG triangles /
quads with SRI


NIPG
SG Q1
DG Triangles
DG Q1 Full
DG Q1 UI3
DG
DG+SRI
log(H1 error)
log(h)


IIPG
SG Q1
DG Triangles
DG Q1 Full
DG Q1 UI3
DG
DG+SRI
log(H1 error)


SIPG
SG Q1
DG Triangles
DG Q1 Full
DG Q1 UI3
DG
DG+SRI
log(H1 error)

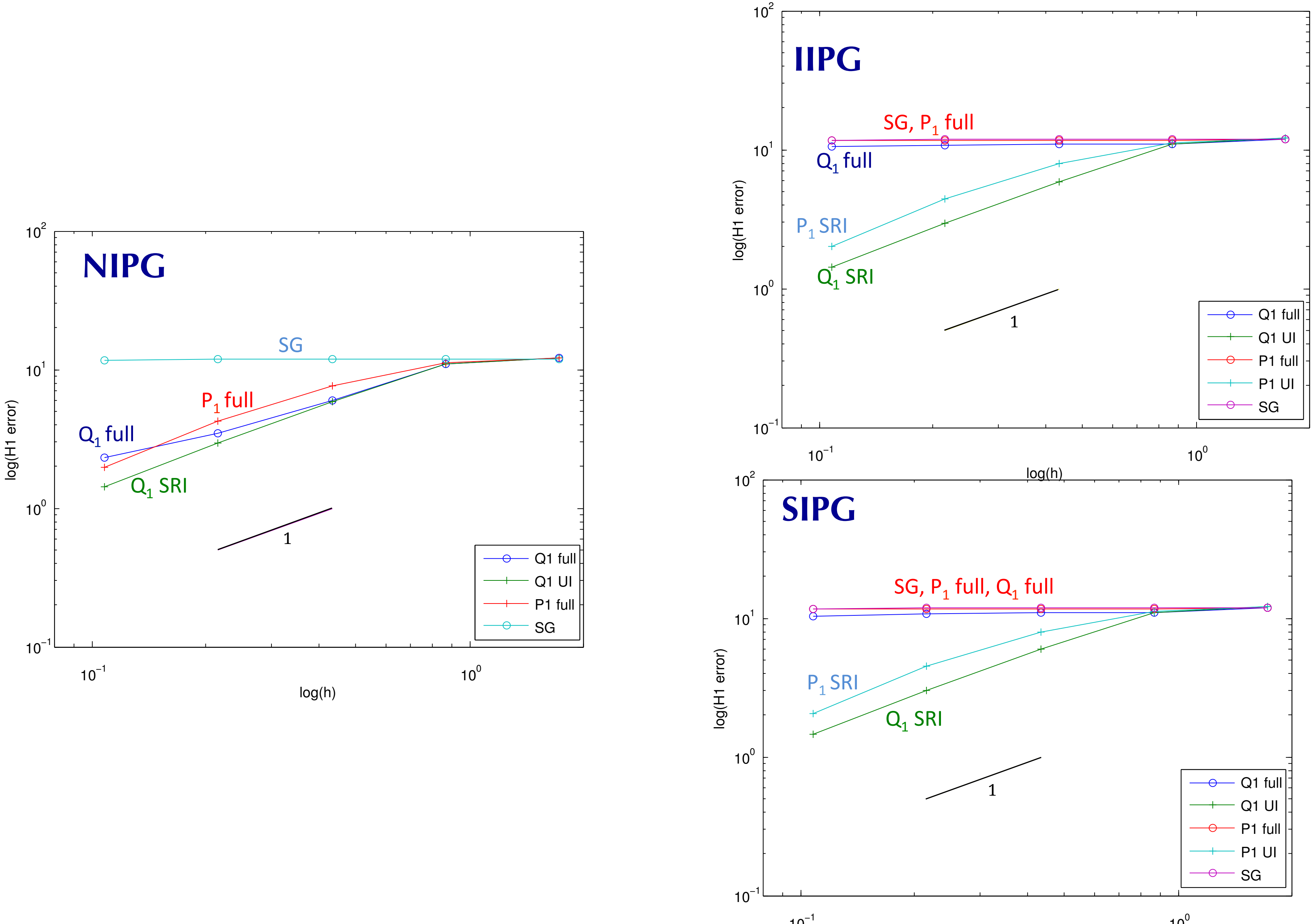
Cube with prescribed body force



$$u_1 = (\cos 2\pi x - 1)(\sin 2\pi y \, \sin \pi z - \sin \pi y \, \sin 2\pi z) + \frac{1}{1+\lambda} \sin \pi x \, \sin \pi y \, \sin \pi z,$$

$$u_2 = (\cos 2\pi y - 1)(\sin 2\pi z \, \sin \pi x - \sin \pi z \, \sin 2\pi x) + \frac{1}{1+\lambda} \sin \pi x \, \sin \pi y \, \sin \pi z,$$

$$u_3 = (\cos 2\pi z - 1)(\sin 2\pi x \, \sin \pi y - \sin \pi x \, \sin 2\pi y) + \frac{1}{1+\lambda} \sin \pi x \, \sin \pi y \, \sin \pi z.$$

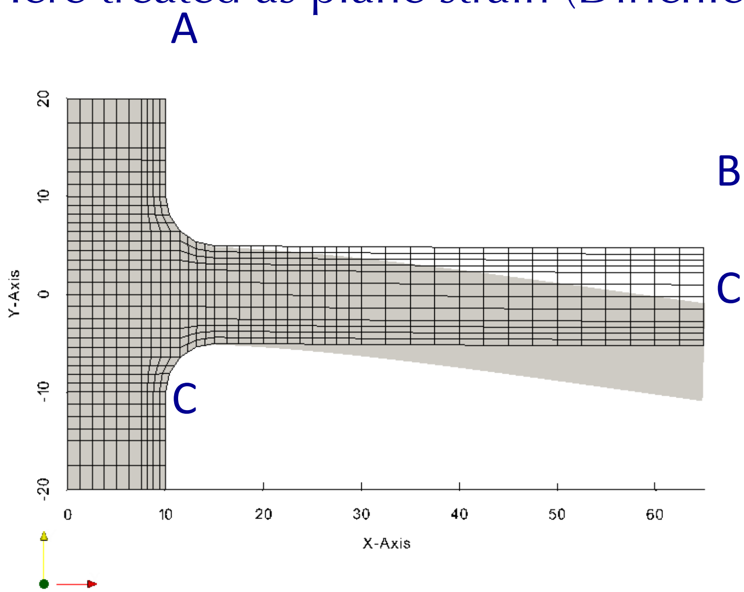

NIPG
SG
P1 full
Q1 full
Q1 SRI
1
Q1 full
Q1 UI
P1 full
SG
log(H1 error)
log(h)
IIPG
SG, P1 full
Q1 full
P1 SRI
Q1 SRI
1
Q1 full
Q1 UI
P1 full
P1 UI
SG
log(H1 error)
log(h)
SIPG
SG, P1 full, Q1 full
P1 SRI
Q1 SRI
1
Q1 full
Q1 UI
P1 full
P1 UI
SG
log(H1 error)

Back to the T-bar example

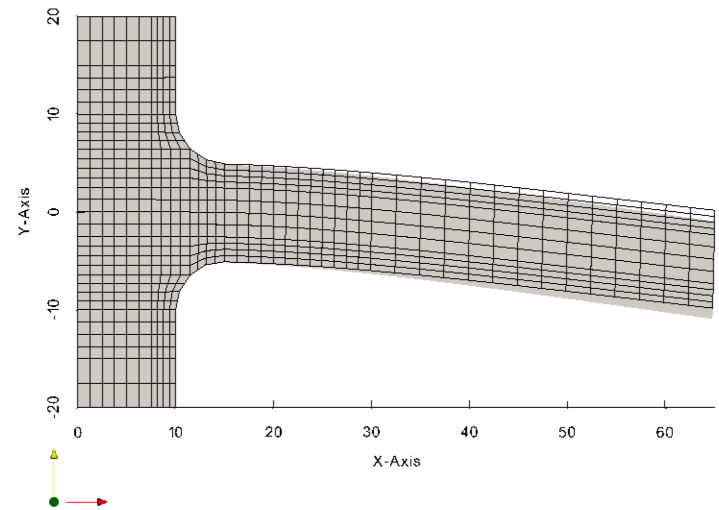
LIU. WHEELER, DAWSON 2009 appeared to show good behaviour for IIPG

3D problem but effectively plane stress (Neumann or flux-free boundary condition) in out-of-plane direction, along ABCD

Here treated as plane strain (Dirichlet or constrained displacement) along ABCD



Without SRI



With SRI

Concluding remarks

- For near-incompressibility DG with quadrilateral elements is not straightforward
- A remedy, viz. selective under-integration λ-dependent edge terms, has been proposed, analysed, and shown to converge uniformly at the optimal rate
- Arbitrary quadrilaterals: numerical experiments indicate behaviour similar to that for rectangles. Analysis would be quite complex
- Current work: nonlinear problems; other parameter-dependent problems

References

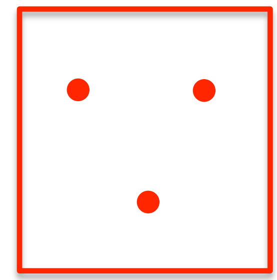
Grieshaber BJ, McBride AT and Reddy BD, Uniformly convergent interior penalty approximations using multilinear approximations for problems in elasticity. *SIAM J. Numer. Anal.* 53 (2015) 2255–2278.

Grieshaber BJ, McBride AT and Reddy BD, Computational aspects of Discontinuous Galerkin approximations using quadrilateral and hexahedral elements. *In review.*

Alternative approach: P_1 approximation on quads

Much of the manipulations carry over, all the bounds are as before, but this time

$$\boldsymbol{\xi} = \Pi \boldsymbol{u} - \boldsymbol{u}_h \in V_h \sim \mathcal{P}_1$$



$\boldsymbol{u}_h \in \mathcal{P}_1$

Problematic terms:

(II) =
$$\theta\lambda \int_E [\![\boldsymbol{\xi}]\!]\{\mathrm{div}\,\boldsymbol{\eta}\}\; ds \;=\; \theta\lambda \sum_E]\{\mathrm{div}\,\boldsymbol{\eta}\} \int_E [\![\boldsymbol{\xi}]\!]\; ds$$

$= 0$ given properties of interpolant

But

(IV) = $k_\lambda \lambda \dfrac{1}{h_E} \displaystyle\int_E [\![\boldsymbol{\xi}]\!]\, [\![\boldsymbol{\eta}]\!]\; ds \neq 0$ so remains a problem for SIPG and IIPG,

but is absent for NIPG

Square plate, DG with quads and P_1