On Multi-Domain Polynomial Interpolation Error Bounds

Samuel Mutua, Prof. S.S. Motsa

University of KwaZulu-Natal
Inyuvesi YakwaZulu-Natali

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Outline

Aim

- To state and prove theorems that govern error bounds in polynomial interpolation.
- To investigate why the Gauss-Lobatto grids points are preferably used in spectral based collocation methods of solution for solving differential equations.
- To highlight on some benefits of multi-domain approach to polynomial interpolation and its application.
- To apply piecewise interpolating polynomial in approximating solution of a differential equation.

Function of one variable

Theorem 1

If $y_N(x)$ is a polynomial of degree at most N that interpolates $y(x)$ at $(N+1)$ distinct grid points $\{x_j\}_{j=0}^{N} \in [a,b]$, and if the first $(N+1)$-th derivatives of $y(x)$ exists and are continuous, then, $\forall x \in [a,b]$ there exist a ξ_x [1] for which

$$E(x) \leq \frac{1}{(N+1)!} y^{(N+1)}(\xi_x) \prod_{j=0}^{N} (x - x_j). \tag{1}$$

Equispaced Grid Points

$$\{x_j\}_{j=0}^{N} = a + jh,\ h = \frac{b-a}{N}$$

Theorem 2

The error bound when equispaced grid points $\{x_j\}_{j=0}^{N} \in [a, b]$, are used in univariate polynomial interpolation is given by

$$E(x) \leq \frac{(h)^{N+1}}{4(N+1)} y^{(N+1)}(\xi_x). \tag{2}$$

Proof

1. Fix x between two grid points, x_k and x_{k+1} so that $x_k \leq x \leq x_{k+1}$ and show that
$$|x - x_k|\,|x - x_{k+1}| \leq \frac{1}{4}h^2.$$
2. The product term $w(x) = \prod_{j=0}^{N}(x - x_j)$ is bounded above by
$$\prod_{j=0}^{N}|x - x_j| \leq \frac{1}{4}h^{N+1}N!.$$
3. Substitute in equation (1) to complete the proof.

Gauss Lobatto (GL) Grid Points

$$\{x_j\}_{j=0}^{N} = \left(\frac{b-a}{2}\right)\cos\left(\frac{j\pi}{N}\right) + \left(\frac{b+a}{2}\right)$$

Theorem 3

The error bound when GL grid points $\{x_j\}_{j=0}^{N} \in [a,b]$, are used in univariate polynomial interpolation is given by

$$E(x) \leq \frac{\left(\frac{b-a}{2}\right)^{N+1}}{K_N(N+1)!} y^{(N+1)}(\xi_x), \tag{3}$$

where

$$K_N = \left(\frac{N}{N+1}\right)^2 \left[\frac{(2N)!}{2^N (N!)^2}\right].$$

Proof

1. The Gauss-Lobatto nodes are roots of the polynomial

$$\begin{aligned} L_{N+1}(\hat{x}) &= (1-\hat{x}^2)P'_N(\hat{x}) \\ &= -N\hat{x}P_N(\hat{x}) + NP_{N-1}(\hat{x}) \\ &= (N+1)\hat{x}P_N(\hat{x}) - (N+1)P_{N+1}(\hat{x}). \end{aligned}$$

2. The polynomial $L_{N+1}(\hat{x})$ in the interval $\hat{x} \in [-1, 1]$ is bounded above by

$$\max_{-1\le\hat{x}\le 1} |L_{N+1}(\hat{x})| \le 2(N+2).$$

3. Express $L_{N+1}(\hat{x})$ as a monic polynomial

$$\frac{L_{N+1}(\hat{x})}{2(N+1)} = \frac{1}{K_N}(\hat{x}-\hat{x}_0)(\hat{x}-\hat{x}_1)\dots(\hat{x}-\hat{x}_N).$$

4. Here

$$K_N = \left(\frac{N}{N+1}\right)^2 \left[\frac{(2N)!}{2^N(N!)^2}\right].$$

5. Substitute in equation (1) to complete the proof.

Chebyshev Grid Points [5]

$$\{x_j\}_{j=0}^{N} = \left(\frac{b-a}{2}\right)\cos\left(\frac{2j+1}{2N+2}\pi\right) + \left(\frac{b+a}{2}\right)$$

Theorem 4

The error bound when Chebyshev grid points $\{x_j\}_{j=0}^{N} \in [a, b]$, are used in univariate polynomial interpolation is given by

$$E(x) \leq \frac{\left(\frac{b-a}{2}\right)^{N+1}}{2^N (N+1)!} y^{(N+1)}(\xi_x). \tag{4}$$

Proof

1. The leading coefficient of $(N+1)$-th degree Chebyshev polynomial is 2^N.
2. Take
$$w(\hat{x}) = \frac{1}{2^N} T_{N+1}(\hat{x}), \quad \text{where} \quad \left| \frac{1}{2^N} T_{N+1}(\hat{x}) \right| \le \frac{1}{2^N},$$
to be the monic polynomial whose roots are the Chebyshev nodes.
3. Substitute in equation (1) to complete the proof.

- We note that for $N > 3$,

$$\frac{\left(\frac{b-a}{N}\right)^{N+1}}{4(N+1)} > \frac{(b-a)^{N+1}}{K_N(2)^{N+1}(N+1)!} > \frac{(b-a)^{N+1}}{2(4)^N(N+1)!}.$$

Function of many variables

Theorem 5

Let $u(x,t) \in C^{N+M+2}([a,b] \times [0,T])$ be sufficiently smooth such that at least the $(N+1)$-th partial derivative with respect to x, $(M+1)$-th partial derivative with respect to t and $(N+M+2)$-th mixed partial derivative with respect to x and t exists and are all continuous, then there exists values $\xi_x, \xi_x' \in (a,b)$, and $\xi_t, \xi_t' \in (0,T)$, [2] such that

$$E(x,t) \leq \frac{\partial^{N+1} u(\xi_x,t)}{\partial x^{N+1}(N+1)!} \prod_{i=0}^{N}(x-x_i) + \frac{\partial^{M+1} u(x,\xi_t)}{\partial t^{M+1}(M+1)!} \prod_{j=0}^{M}(t-t_j) - \frac{\partial^{N+M+2} u(\xi_x',\xi_t')}{\partial x^{N+1}\partial t^{M+1}(N+1)!(M+1)!} \prod_{i=0}^{N}(x-x_i) \prod_{j=0}^{M}(t-t_j). \tag{5}$$

Equispaced

Theorem 6

The error bound when equispaced grid points $\{x_i\}_{i=0}^{N} \in [a,b]$ and $\{t_j\}_{j=0}^{M} \in [0,T]$, in x-variable and t-variable, respectively, are used in bivariate polynomial interpolation is given by

$$\begin{aligned} E(x,t) = |u(x,t) - U(x,t)| \leq & C_1 \frac{\left(\frac{b-a}{N}\right)^{N+1}}{4(N+1)} + C_2 \frac{\left(\frac{T}{M}\right)^{M+1}}{4(M+1)} \\ & + C_3 \frac{\left(\frac{b-a}{N}\right)^{N+1} \left(\frac{T}{M}\right)^{M+1}}{4^2(N+1)(M+1)}. \end{aligned} \tag{6}$$

Gauss Lobatto

Theorem 7

The error bound when GL grid points $\{x_i\}_{i=0}^{N} \in [a, b]$, in x-variable and $\{t_j\}_{j=0}^{M} \in [0, T]$, in t-variable are used in bivariate polynomial interpolation is given by

$$
\begin{aligned}
E(x,t) \leq C_1 \frac{(b-a)^{N+1}}{2^{N+1} K_N (N+1)!} + C_2 \frac{(T)^{M+1}}{2^{M+1} K_M (M+1)!} \\
+ C_3 \frac{(b-a)^{N+1} (T)^{M+1}}{(2)^{(N+M+2)} K_N K_M (N+1)!(M+1)!},
\end{aligned} \tag{7}
$$

where

$$
K_N = \left(\frac{N}{N+1}\right)^2 \left(\frac{(2N)!}{2^N (N!)^2}\right).
$$

Chebyshev

Theorem 8

The error bound for Chebyshev grid points $\{x_i\}_{i=0}^{N} \in [a, b]$ and $\{t_j\}_{j=0}^{M} \in [0, T]$, in x-variable and t-variable, respectively, in bivariate polynomial interpolation is given by

$$E(x,t) \leq C_1 \frac{(b-a)^{N+1}}{2(4)^N (N+1)!} + C_2 \frac{(T)^{M+1}}{2(4)^M (M+1)!} + C_3 \frac{(b-a)^{N+1}(T)^{M+1}}{2^2(4)^{N+M}(N+1)!(M+1)!}. \tag{8}$$

Generalized multi-variate polynomial interpolation

If $U(x_1, x_2, \ldots, x_n)$ approximates $u(x_1, x_2, \ldots, x_n)$, $(x_1, x_2, \ldots, x_n) \in [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$, and suppose that there are $N_i,\ i = 1, 2, \ldots, n$ grid points in x_i-variable, then the error bound in the best approximation is

$$\begin{aligned} E_c \leq & C_1 \frac{(b_1 - a_1)^{N_1+1}}{2(4)^{N_1}(N_1+1)!} + C_2 \frac{(b_2 - a_2)^{N_2+1}}{2(4)^{N_2}(N_2+1)!} \\ & + \ldots + C_n \frac{(b_n - a_n)^{N_n+1}}{2(4)^{N_n}(N_n+1)!} \\ & + C_{n+1} \frac{(b_1 - a_1)^{N_1+1}(b_2 - a_2)^{N_2+1} \ldots (b_n - a_n)^{N_n+1}}{2^n(4)^{(N_1+N_2+\ldots+N_n)}(N_1+1)!(N_2+1)! \ldots (N_n+1)!}. \end{aligned} \tag{9}$$

$$C_{n+1} = \max_{[x_1, x_2, \ldots, x_n] \in \Omega} \left| \frac{\partial^{(N_1+N_2+\ldots+N_n+n)} u(x_1, x_2, x_3, \ldots, x_n)}{\partial x_1^{N_1+1} \partial x_2^{N_2+1} \ldots \partial x_n^{N_n+1}} \right|. \tag{10}$$

Illustration of the concept of multi-domain [3]

- Let $t \in \Gamma$ where $\Gamma \in [0, T]$. The domain Γ is decomposed into p non-overlapping subintervals as

$$\Gamma_k = [t_{k-1}, t_k],\ t_{k-1} < t_k,\ t_0 = 0,\ t_p = T,\ k = 1, 2, \ldots, p.$$

STRATEGY

- Perform interpolation on each subinterval.
- Define the interpolating polynomial over the entire domain in piece-wise form.

Equispaced

Theorem 9

The error bound when equispaced grid points $\{x_i\}_{i=0}^{N} \in [a, b]$ for x-variable and $\{t_j^{(k)}\}_{j=0}^{M} \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, p$, for the decomposed domain in t-variable, are used in bivariate polynomial interpolation is given by

$$\begin{aligned} E(x,t) \leq & C_1 \frac{\left(\frac{b-a}{N}\right)^{N+1}}{4(N+1)} + \left(\frac{1}{p}\right)^{M} C_2 \frac{\left(\frac{T}{M}\right)^{M+1}}{4(M+1)} \\ & + \left(\frac{1}{p}\right)^{M} C_3 \frac{\left(\frac{b-a}{N}\right)^{N+1}\left(\frac{T}{M}\right)^{M+1}}{4^2(N+1)(M+1)}. \end{aligned} \tag{11}$$

Proof

- Each subinterval

$$\left|\prod_{j=0}^{M}(t-t_j^{(k)})\right| \le \frac{1}{4}\left(\frac{T}{pM}\right)^{M+1} M! = \left(\frac{1}{p}\right)^{M+1}\frac{1}{4}\left(\frac{T}{M}\right)^{M+1} M!.$$

- Break $C_2\frac{\left(\frac{T}{M}\right)^{M+1}}{4(M+1)}$ into $\sum_{k=1}^{p}\left(\frac{1}{p}\right)^{M+1} C_2^{(k)}\frac{\left(\frac{T}{M}\right)^{M+1}}{4(M+1)}$.

  where

$$\max_{(x,t)\in\Omega}\left|\frac{\partial^{M+1}u(x,t)}{\partial t^{M+1}}\right| = \left|\frac{\partial^{M+1}u(x,\xi_k)}{\partial t^{M+1}}\right| \le C_2^{(k)}, \quad t\in[t_{k-1},t_k].$$

- Multi-Domain

$$\sum_{k=1}^{p}\left(\frac{1}{p}\right)^{M+1} C_2^{(k)}\frac{\left(\frac{T}{M}\right)^{M+1}}{4(M+1)} \le \left(\frac{1}{p}\right)^{M} C_2\frac{\left(\frac{T}{M}\right)^{M+1}}{4(M+1)}. \tag{12}$$

- Similarly, last term in equation (6) reduces to $\left(\frac{1}{p}\right)^{M} C_3\frac{\left(\frac{b-a}{N}\right)^{N+1}\left(\frac{T}{M}\right)^{M+1}}{4^2(N+1)(M+1)}$.

Gauss Lobatto

Theorem 10

The error bound when Gauss-Lobatto grid points $\{x_i\}_{i=0}^{N} \in [a, b]$ for x-variable and $\{t_j^{(k)}\}_{j=0}^{M} \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, p$, for the decomposed domain in t-variable, are used in bivariate polynomial interpolation is given by

$$E(x,t) \leq C_1 \frac{(b-a)^{N+1}}{2^{N+1} K_N (N+1)!} + \left(\frac{1}{p}\right)^{M} C_2 \frac{(T)^{M+1}}{2^{M+1} K_M (M+1)!} + \left(\frac{1}{p}\right)^{M} C_3 \frac{(b-a)^{N+1} (T)^{M+1}}{(2)^{(N+M+2)} K_N K_M (N+1)!(M+1)!}. \tag{13}$$

Chebyshev

Theorem 11

The error bound when Chebyshev grid points $\{x_i\}_{i=0}^{N} \in [a, b]$ for x-variable and $\{t_j^{(k)}\}_{j=0}^{M} \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, P$ for the decomposed domain in t-variable, are used in bivariate polynomial interpolation is given by

$$
\begin{aligned}
E(x,t) \leq\, & C_1 \frac{(b-a)^{N+1}}{2(4)^N (N+1)!} + \left(\frac{1}{p}\right)^M C_2 \frac{(T)^{M+1}}{2(4)^M (M+1)!} \\
& + \left(\frac{1}{p}\right)^M C_3 \frac{(b-a)^{N+1} (T)^{M+1}}{2^2 (4)^{\{N+M\}} (N+1)!(M+1)!}.
\end{aligned} \tag{14}
$$

Test Example

Example

Consider the Burgers-Fisher equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad x \in (0,5), \quad t \in (0,10], \tag{15}$$

subject to boundary conditions

$$u(0,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{5t}{8}\right), \; u(5,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{5t}{8} - \frac{5}{4}\right), \tag{16}$$

and initial condition

$$u(x,0) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{4}\right). \tag{17}$$

The exact solution given in [4] as

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{5t}{8} - \frac{x}{4}\right). \tag{18}$$

Single VS Multiple domains

Table: 2: Absolute error values $N = 20\ M = 50$ **Single**, $N = 20\ M = 10\ p = 5$ **Multiple**

	Single Domain		Multi- Domain	
$x \backslash t$	5.0	10.0	5.0	10.0
0.4775	2.0474e-009	6.2515e-012	5.0959e-014	4.9849e-014
1.3650	4.8463e-009	3.0746e-011	1.0880e-014	9.9920e-015
2.5000	7.8205e-009	8.6617e-012	1.2546e-014	3.5527e-015
3.6350	1.8239e-008	3.6123e-010	1.1102e-015	4.1078e-015
4.5225	1.9871e-008	6.3427e-0010	6.4060e-014	1.1768e-014
CPU Time	2.132547 sec		0.018469 sec	
Cond NO	6.3710e004		3.3791e003	
Matrix D	1000×1000		200×200, 5 times	

Theoretical VS Numerical

Table: 1: Comparison of theoretical values of error bounds with the numerical values.

N	Error	Equispaced	Gauss-Lobatto	Chebyshev
2*5	Bound	1.2288×10^{-1}	4.9887×10^{-2}	3.1250×10^{-2}
	Numerical	1.4091×10^{-2}	1.0772×10^{-2}	8.1343×10^{-3}
2*10	Bound	1.6893×10^{-2}	1.4519×10^{-3}	8.8794×10^{-4}
	Numerical	7.9134×10^{-4}	7.0721×10^{-5}	6.1583×10^{-5}
2*20	Bound	5.7644×10^{-4}	2.0355×10^{-6}	9.0383×10^{-7}
	Numerical	5.8480×10^{-6}	1.0942×10^{-8}	9.1555×10^{-9}

The function considered is $f(x) = \frac{1}{1+x^2}$.

Conclusion

1. Although Gauss-Lobatto nodes yield larger interpolation error than Chebyshev nodes the difference is negligible.
2. Gauss-Lobatto nodes are preferred to Chebyshev nodes when solving differential equations using spectral collocation based methods as they are convenient to use.
3. Multi-domain application:
   - Approximating functions: Unbounded higher ordered derivative, or those that do not possess higher ordered derivatives.
   - Approximating the solution of differential equations that are defined over large domains.

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