Simultaneous Gaussian quadrature for Angelesco systems

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[1] Joint work with Doron Lubinsky

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r measures $\mu_1, \ldots, \mu_r$ are given, one function $f : \mathbb{R} \to \mathbb{R}$.
We want to approximate r integrals

$$\int f(x)\, d\mu_j(x), \qquad 1 \leq j \leq r$$

by means of quadrature sums of the form

$$\sum_{k=1}^{N} \lambda_{k,N}^{(j)} f(x_{k,N}), \qquad 1 \leq j \leq r,$$

and the quadrature needs to be correct for polynomials of degree as high as possible.

Multiple orthogonal polynomials

Type II multiple orthogonal polynomial $P_{\vec{n}}$ for the multi-index $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ is the monic polynomial of degree $|\vec{n}| = n_1 + n_2 + \ldots + n_r$ for which

$$\int x^k P_{\vec{n}}(x)\, d\mu_j(x) = 0, \qquad 0 \leq k \leq n_j - 1, \quad 1 \leq j \leq r.$$

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Theorem

If we choose as nodes the zeros $x_{k,\vec{n}}$ of $P_{\vec{n}}$ and if we take

$$\lambda_{k,\vec{n}}^{(j)} = \int \ell_{k,\vec{n}}(x)\, d\mu_j(x),$$

($\ell_{k,\vec{n}}$ fundamental polynomials of Lagrange interpolation). Then

$$\sum_{j=1}^{|\vec{n}|} \lambda_{k,\vec{n}}^{(j)} p(x_{k,\vec{n}}) = \int p(x)\, d\mu_j(x), \qquad 1 \leq j \leq r$$

whenever p is a polynomial of degree at most $|\vec{n}| + n_j - 1$.

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If we choose as nodes the zeros $x_{k,rn}$ of $P_{(n,n,\ldots,n)}$ and if we take

$$\lambda_{k,rn}^{(j)} = \int \ell_{k,rn}(x)\, d\mu_j(x),$$

($\ell_{k,rn}$ fundamental polynomials of Lagrange interpolation). Then

$$\sum_{j=1}^{rn} \lambda_{k,rn}^{(j)} p(x_{k,rn}) = \int p(x)\, d\mu_j(x), \qquad 1 \leq j \leq r$$

whenever p is a polynomial of degree at most $(r+1)n - 1$.

Angelesco systems

An Angelesco system is a system of r measures such that

$$\mathrm{supp}(\mu_j) \subset [a_j, b_j]$$

and the intervals $(a_1, b_1), \ldots, (a_r, b_r)$ are disjoint.

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The type II multiple orthogonal polynomial $P_{\vec{n}}$ for an Angelesco system always exists and has n_j zeros on (a_j, b_j) for $1 \leq j \leq r$:

$$P_{\vec{n}}(x) = \prod_{j=1}^{r} p_{\vec{n},j}(x).$$

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In fact $p_{\vec{n},j}$ is an ordinary orthogonal polynomial of degree n_j on $[a_j, b_j]$ for the measure $\prod_{i \neq j} |p_{\vec{n},i}(x)| \, d\mu_j(x)$.

Known results

Property (Nikishin-Sorokin)

The quadrature weights have the following properties

$$\lambda_{k,rn}^{(j)} > 0, \qquad \text{if } x_{k,rn} \in [a_j, b_j],$$

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and $\lambda_{k,rn}^{(j)}$ *has alternating sign when* $x_{k,rn} \in [a_i, b_i]$ *with* $i \neq j$*. Furthermore* $\lambda_{k,rn}^{(j)}$ *is positive for the zero in* $[a_i, b_i]$ *closest to* $[a_j, b_j]$*.*

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We consider only two intervals $[a_1, b_1]$ and $[a_2, b_2]$, with $b_1 \leq a_2$.

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$$P_{(n,n)}(x) = (-1)^n p_n(x) q_n(x),$$

with

$$p_n(x) = \prod_{j=1}^{n} (x - x_{j,2n}), \quad q_n(x) = (-1)^n \prod_{j=n+1}^{2n} (x - x_{j,2n}),$$

i.e., p_n has zeros on $[a_1, b_1]$ and q_n has zeros on $[a_2, b_2]$.

Simultaneous quadrature on two intervals

Quadrature rules:

$$\sum_{j=1}^{2n} \lambda_{j,2n}^{(1)} P(x_{j,2n}) = \int_{a_1}^{b_1} P(x)\, d\mu_1(x),$$

$$\sum_{j=1}^{2n} \lambda_{j,2n}^{(2)} P(x_{j,2n}) = \int_{a_2}^{b_2} P(x)\, d\mu_2(x),$$

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Choose $P(x) = \pi(x) q_n(x)$, then

$$\sum_{j=1}^{n} \lambda_{j,2n}^{(1)} \pi(x_{j,2n}) q_n(x_{j,2n}) = \int_{a_1}^{b_1} \pi(x)\; q_n(x)\, d\mu_1(x),$$

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$\lambda_{j,2n}^{(1)} q_n(x_{j,2n}) = \lambda_{j,n}(q_n\, d\mu_1), \quad 1 \leq j \leq n,$ Gauss quadrature

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Interpolatory quadrature formula for $[a_1, b_1]$ with quadrature nodes on $[a_2, b_2]$

Possé-Chebyshev-Markov-Stieltjes inequalities

Theorem (Lubinsky-WVA)

Suppose $1 \leq \ell \leq n$ *and* $g : (-\infty, x_{\ell,2n}] \to [0,\infty)$ *has* $2n$ *continuous derivatives, with*

$$g^{(k)}(x) \geq 0, \qquad 0 \leq k \leq 2n.$$

Then

$$\sum_{k=1}^{\ell-1} \lambda_{k,2n}^{(1)} g(x_{k,2n}) \leq \int_{a_1}^{x_{\ell,2n}} g(x)\, d\mu_1(x) \leq \sum_{k=1}^{\ell} \lambda_{k,2n}^{(1)} g(x_{k,2n}).$$

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Follows from the Possé-Chebyshev-Markov-Stieltjes inequalities for the measure $q_n\, d\mu_1$ and the fact that g/q_n is completely monotonic on $[a_1, b_1]$.

Chebyshev-Markov-Stieltjes inequalities

Property

For $1 \leq \ell \leq n-1$

$$\lambda_{\ell,2n}^{(1)} \leq \int_{x_{\ell-1,2n}}^{x_{\ell+1,2n}} d\mu_1(x), \quad \lambda_{\ell,2n}^{(1)} + \lambda_{\ell+1,2n}^{(1)} \geq \int_{x_{\ell,2n}}^{x_{\ell+1,2n}} d\mu_1(x),$$

and

$$\sum_{k=1}^{n} \lambda_{k,2n}^{(1)} \leq \int_{a_1}^{b_1} d\mu_1(x).$$

Chebyshev-Markov-Stieltjes inequalities

Property

For $n+1 \leq \ell \leq 2n-1$

$$\lambda_{\ell,2n}^{(2)} \leq \int_{x_{\ell-1,2n}}^{x_{\ell+1,2n}} d\mu_2(x), \quad \lambda_{\ell,2n}^{(2)} + \lambda_{\ell+1,2n}^{(2)} \geq \int_{x_{\ell,2n}}^{x_{\ell+1,2n}} d\mu_2(x),$$

and

$$\sum_{k=n+1}^{2n} \lambda_{k,2n}^{(2)} \leq \int_{a_2}^{b_2} d\mu_2(x).$$

Potential theory

Suppose that $\mu_1' > 0$ a.e. on $[a_1, b_2]$ and $\mu_2' > 0$ a.e. on $[a_2, b_2]$.

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asymptotic distribution of the zeros of p_n:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(x_{j,2n}) = \int_{a_1}^{b^*} f(x)\, d\nu_1(x), \qquad f \in C([a_1, b_1])$$

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asymptotic distribution of the zeros of q_n:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=n+1}^{2n} g(x_{j,2n}) = \int_{a^*}^{b_2} g(x)\, d\nu_2(x), \qquad g \in C([a_2, b_2])$$

Potential theory

The limiting measures (ν_1, ν_2) satisfy a **vector equilibrium problem**

$$I(\nu_1) + I(\nu_2) + I(\nu_1, \nu_2) = \min\Big(I(\tau_1) + I(\tau_2) + I(\tau_1, \tau_2) \Big)$$

over all probability measures τ_1 on $[a_1, b_1]$ and τ_2 on $[a_2, b_2]$, where

$$I(\tau_i, \tau_j) = \int_{a_i}^{b_i} \int_{a_j}^{b_j} \log \frac{1}{|x-y|} \, d\tau_j(x) \, d\tau_i(y),$$

and $I(\tau_i) = I(\tau_i, \tau_i)$.

Potential theory

Variational conditions for the potentials of ν_1 and ν_2

$$U(x;\nu_1) = \int \log \frac{1}{|x-y|}\, d\nu_1(y), \quad U(x;\nu_2) = \int \log \frac{1}{|x-y|}\, d\nu_2(y).$$

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$$2U(x;\nu_1) + U(x;\nu_2) \begin{cases} = \ell_1, & x \in [a_1, b^*], \\ > \ell_1, & x \in (b^*, b_1], \end{cases}$$

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$$U(x;\nu_1) + 2U(x;\nu_2) \begin{cases} = \ell_2, & x \in [a^*, b_2], \\ > \ell_2, & x \in [a_2, a^*). \end{cases}$$

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- $b^* = b_1$ and $a^* = a_2$
- $b^* < b_1$ and $a^* = a_2$
- $b^* = b_1$ and $a_2 < a^*$

three possibilities

Potential theory

$$\frac{1}{\gamma_n^2(q_n\, d\mu_1)} = \int_{a_1}^{b_1} p_n^2(x)\ q_n(x)\, d\mu_1(x),$$

$$\frac{1}{\gamma_n^2(p_n\, d\mu_2)} = \int_{a_2}^{b_2} q_n^2(x)\ p_n(x)\, d\mu_2(x)$$

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Asymptotic behavior of these norms

$$\lim_{n\to\infty} \gamma_n(q_n\,d\mu_1)^{1/n} = e^{\ell_1/2}, \quad \lim_{n\to\infty} \gamma_n(p_n\,d\mu_2)^{1/n} = e^{\ell_2/2}.$$

Gonchar-Rakhmanov 1981, Nikishin-Sorokin 1988/1991

Estimates

Property

For $1 \leq j \leq n$ one has

$$\lambda_{j,2n}^{(1)} \geq \lambda_m(x_{j,2n};\mu_1), \qquad m = \lceil \frac{3n}{2} \rceil.$$

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$$\lambda_{j,2n}^{(1)} \geq \lambda_m(x_{j,2n}; \mu_1), \qquad m = \lceil \frac{3n}{2} \rceil.$$

If J_1 is a closed subinterval of (a_1, b_1) and μ_1 is absolutely continuous in an open neighborhood of J_1, while μ_1' is bounded from below by a positive constant there, then

$$\lambda_{j,2n}^{(1)} \geq \frac{C_1}{n}, \qquad x_{j,2n} \in J_1.$$

Estimates

Property

Suppose $b_1 < a_2$. Then uniformly on compact subsets of (a_1, b^*)

$$\lambda_{j,2n}^{(1)} \leq \bigl(1 + o(1)\bigr) \lambda_{\lfloor n/2 \rfloor}(x_{j,2n}; \mu_1).$$

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Convergence results: positive weights

Theorem (Lubinsky-WVA)

Let f be a continuous function on $[a_1, b_1]$ and $f(b^*) = 0$, with $[a_1, b^*] = \mathrm{supp}(\nu_1)$. Then

$$\lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{j,2n}^{(1)} f(x_{j,2n}) = \int_{a_1}^{b^*} f(x)\, d\mu_1(x).$$

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The first quadrature rule converges for functions f that can be approximated by **weighted** polynomials (weight $q_n(x)$), i.e., there exists a sequence of polynomials $(R_n)_n$ such that

$$\lim_{n\to\infty} \|f - R_n q_n\|_{[a_1,b_1]} = 0.$$

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These are continuous functions that vanish outside (a_1, b^*) [Totik, 1994].

Convergence results: positive weights

Theorem (Lubinsky-WVA)

Let g be a continuous function on $[a_2, b_2]$ and $g(a^*) = 0$, with $[a^*, b_2] = \mathrm{supp}(\nu_2)$. Then

$$\lim_{n\to\infty} \sum_{j=n+1}^{2n} \lambda_{j,2n}^{(2)} g(x_{j,2n}) = \int_{a^*}^{b_2} g(x)\, d\mu_2(x).$$

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Convergence results: alternating weights

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Suppose that $\mu_1' > 0$ *on* $[a_1, b_1]$ *and* $\mu_2' > 0$ *on* $[a_2, b_2]$*, and* $b_1 < a_2$*. Then*

$$\lim_{n\to\infty} |\lambda_{j,2n}^{(1)}|^{1/n} = \exp\Big(2U(x;\nu_1) + U(x;\nu_2) - \ell_1\Big),$$

whenever $x_{j,2n} \to x \in [a^*, b_2]$*.*

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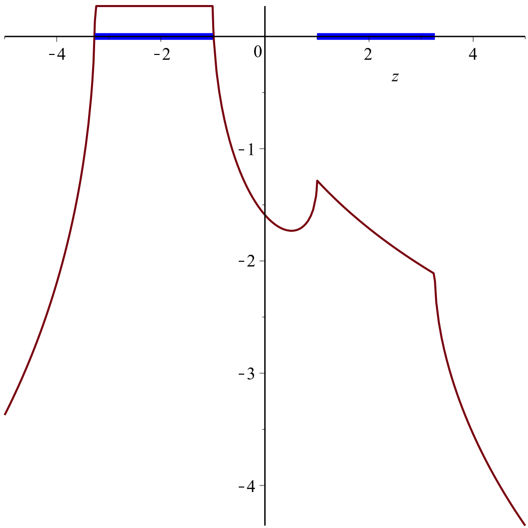
whenever $x_{j,2n} \to x \in [a^*, b_2]$.

The weights $|\lambda_{j,2n}^{(1)}|$ ($n+1 \leq j \leq 2n$) grow exponentially fast if $2U(x;\nu_1) + U(x;\nu_2) > \ell_1$

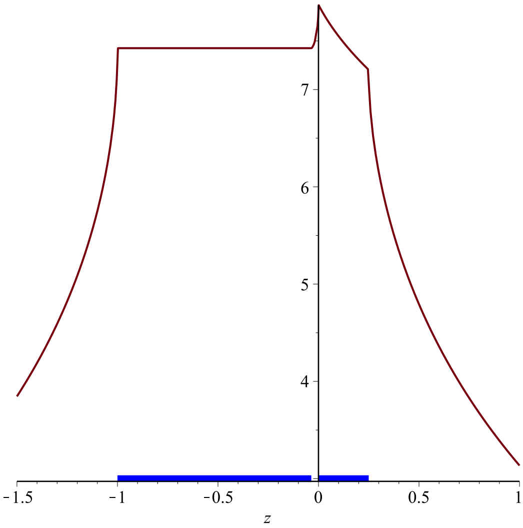
The weights decrease to zero exponentially fast if $2U(x;\nu_1) + U(x;\nu_2) < \ell_1$.

Example: two equal length intervals



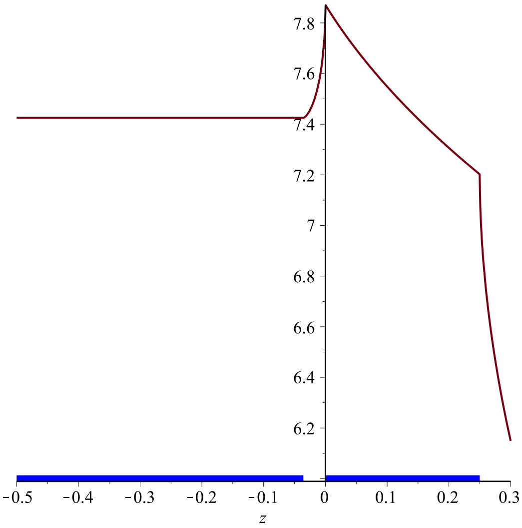
The function $2U(z;\nu_1) + U(z;\nu_2)$ for $[-3.26, -1]$ and $[1, 3.26]$

Example: two intervals of different length



The function $2U(z;\nu_1) + U(z;\nu_2)$ for $[-1,0]$ and $[0,\frac{1}{4}]$

Example: two intervals of different length



The function $2U(z;\nu_1) + U(z;\nu_2)$ for $[-1, 0]$ and $[0, \frac{1}{4}]$ (detail)

Convergence result: equal length intervals

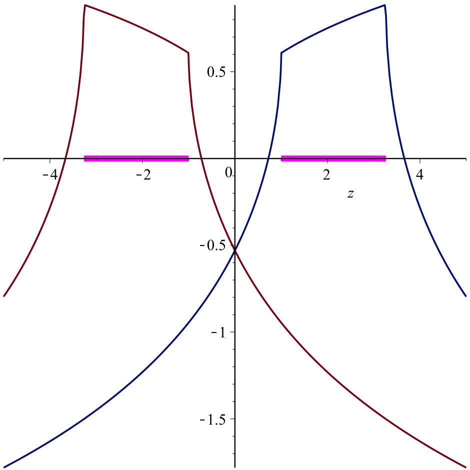
Theorem (Lubinsky-WVA)

Suppose both intervals $[a_1, b_1]$ *and* $[a_2, b_2]$ *are of the same size and let* $b_1 < a_2$*. If* f *is continuous on* $[a_1, b_1]$ *and* $[a_2, b_2]$*, then*

$$\lim_{n\to\infty} \sum_{j=1}^{2n} \lambda_{j,2n}^{(1)} f(x_{j,2n}) = \int_{a_1}^{b_1} f(x)\, d\mu_1(x),$$

$$\lim_{n\to\infty} \sum_{j=1}^{2n} \lambda_{j,2n}^{(2)} f(x_{j,2n}) = \int_{a_2}^{b_2} f(x)\, d\mu_2(x).$$

Example: two intervals of equal length



The potentials $U(z;\nu_1)$ and $U(z;\nu_2)$

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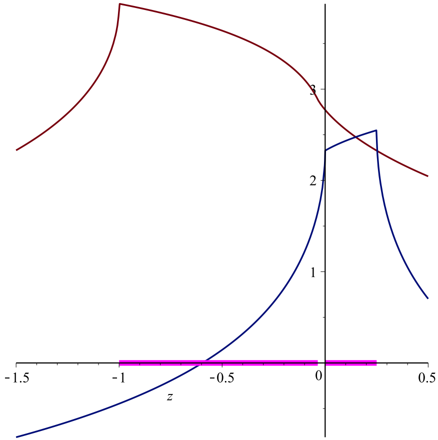

The potentials $U(z;\nu_1)$ and $U(z;\nu_2)$

Hermite-Padé approximation

Let

$$g_1(z) = \int_{a_1}^{b_1} \frac{d\mu_1(x)}{z-x}, \quad g_2(z) = \int_{a_2}^{b_2} \frac{d\mu_2(x)}{z-x}.$$

Hermite-Padé approximation to (g_1, g_2):

$$g_1(z)P_{n,n}(z) - Q_{2n-1}(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right),$$

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Then

$$\frac{Q_{2n-1}(z)}{P_{n,n}(z)} = \sum_{j=1}^{2n} \frac{\lambda_{j,2n}^{(1)}}{z - x_{j,2n}}, \quad \frac{R_{2n-1}(z)}{P_{n,n}(z)} = \sum_{j=1}^{2n} \frac{\lambda_{j,2n}^{(2)}}{z - x_{j,2n}}.$$

Hermite-Padé approximation

$$\sum_{j=1}^{2n} \lambda_{j,2n}^{(1)} f(x_{j,2n}) = \frac{1}{2\pi i} \int_\Gamma f(z) \frac{Q_{2n-1}(z)}{P_{n,n}(z)}\, dz,$$

$$\sum_{j=1}^{2n} \lambda_{j,2n}^{(2)} f(x_{j,2n}) = \frac{1}{2\pi i} \int_\Gamma f(z) \frac{R_{2n-1}(z)}{P_{n,n}(z)}\, dz,$$

where Γ is a closed contour around $[a_1, b_1] \cup [a_2, b_2]$.

Convergence for analytic functions

Theorem

If f is analytic in a domain Ω that contains $C_\gamma^1 = \{z \in \mathbb{C} : 2U(z;\nu_1) + U(z;\nu_2) - \ell_1 > \gamma\}$ with $\gamma < 0$, then

$$\limsup_{n\to\infty} \left| \sum_{j=1}^{2n} \lambda_{j,2n}^{(1)} f(x_{j,2n}) - \int_{a_1}^{b_1} f(x)\, d\mu_1(x) \right|^{1/n} \leq e^{\gamma}.$$

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If f is analytic in a domain Ω that contains $C_\gamma^2 = \{z \in \mathbb{C} : U(z;\nu_1) + 2U(z;\nu_2) - \ell_2 > \gamma\}$ *with* $\gamma < 0$, *then*

$$\limsup_{n\to\infty} \left| \sum_{j=1}^{2n} \lambda_{j,2n}^{(2)} f(x_{j,2n}) - \int_{a_2}^{b_2} f(x)\, d\mu_2(x) \right|^{1/n} \leq e^\gamma.$$

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- Not recommended for Angelesco systems with intervals of different size.
- More useful for measures having the same support, e.g.,

$$\int_{-\infty}^{\infty} f(x)e^{-x^2+c_1x}\,dx,\ \int_{-\infty}^{\infty} f(x)e^{-x^2+c_2x}\,dx,\ \int_{-\infty}^{\infty} f(x)e^{-x^2+c_3x}\,dx$$

  where $c_i/2$ are the wavelengths for red, blue, green and f is a light signal.

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