Robust control in multidimensional systems

Sanne ter Horst [1]
North-West University

SANUM 2016
Stellenbosch University

Joint work with J.A. Ball

NORTH-WEST UNIVERSITY ®
YUNIBESITI YA BOKONE-BOPHIRIMA
NOORDWES-UNIVERSITEIT

[1] This work is based on the research supported in part by the National Research Foundation of South Africa (Grant Numbers 93039, and 93406).

Discrete time linear systems (1D case)

Consider the discrete time linear system

$$\Sigma := \begin{cases} x(n+1) &= Ax(n) + B_1 w(n) + B_2 u(n) \\ v(n) &= C_1 x(n) + D_{11} w(n) + D_{12} u(n) \\ y(n) &= C_2 x(n) + D_{21} w(n) \end{cases} \qquad (n \in \mathbb{Z}_+)$$

with system matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \in \mathbb{C}^{(n+s+p)\times(n+r+q)}.$$

and transfer function

$$G(\lambda) = D + \lambda C(I - \lambda A)^{-1} B.$$

Discrete time linear systems (1D case)

Consider the discrete time linear system

$$\Sigma := \begin{cases} x(n+1) &= Ax(n) + B_1 w(n) + B_2 u(n) \\ v(n) &= C_1 x(n) + D_{11} w(n) + D_{12} u(n) \\ y(n) &= C_2 x(n) + D_{21} w(n) \end{cases} \qquad (n \in \mathbb{Z}_+)$$

with system matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \in \mathbb{C}^{(n+s+p)\times(n+r+q)}.$$

and transfer function

$$G(\lambda) = D + \lambda C(I - \lambda A)^{-1} B.$$

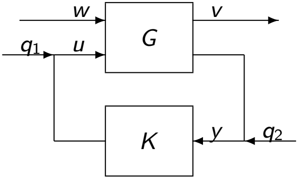
Taking $x(0) = 0$, the Z-transforms $\hat{w}(z) = \sum w(n) z^n$, $\hat{u}$, $\hat{v}$, $\hat{y}$ are related through

$$\begin{bmatrix} \hat{v}(z) \\ \hat{y}(z) \end{bmatrix} = G(z) \begin{bmatrix} \hat{w}(z) \\ \hat{u}(z) \end{bmatrix}$$

Want:
Stability (G analytic in $\mathbb{D} = \{z \colon |z| < 1\}$) & Performance ($\sup_{z\in\mathbb{D}} \|G(z)\| \le 1$).

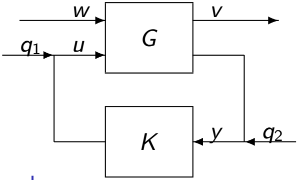
Closed loop system (1D case)

No stability & Performance: Find a controller $K \sim \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ so that the closed loop system Σ_{cl} with transfer function G_{cl} has stability and/or performance:



Closed loop system (1D case)

No stability & Performance: Find a controller $K \sim \left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix}\right]$ so that the closed loop system Σ_{cl} with transfer function G_{cl} has stability and/or performance:



Frequency domain approach

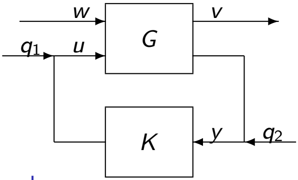
Assume a stabilizing controller K exists and let G_{22} have double co-prime factorization

$$G_{22} = M^{-1}N = \widetilde{N}\widetilde{M}^{-1} \quad \begin{bmatrix} M & -N \\ -\widetilde{Q} & \widetilde{P} \end{bmatrix} \begin{bmatrix} P & \widetilde{N} \\ Q & \widetilde{M} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Then all stabilizing solutions are of the form $K = (Q + \widetilde{M}\Lambda)(P + \widetilde{N}\Lambda)^{-1}$ with Λ stable, $\sup_{z\in\mathbb{D}} \|\Lambda(z)\| < \infty$ s.t. $\det(P + \widetilde{N}\Lambda) \neq 0$.

Closed loop system (1D case)

No stability & Performance: Find a controller $K \sim \left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix}\right]$ so that the closed loop system Σ_{cl} with transfer function G_{cl} has stability and/or performance:



Frequency domain approach

Assume a stabilizing controller K exists and let G_{22} have double co-prime factorization

$$G_{22} = M^{-1}N = \widetilde{N}\widetilde{M}^{-1} \quad \begin{bmatrix} M & -N \\ -\widetilde{Q} & \widetilde{P} \end{bmatrix} \begin{bmatrix} P & \widetilde{N} \\ Q & \widetilde{M} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Then all stabilizing solutions are of the form $K = (Q + \widetilde{M}\Lambda)(P + \widetilde{N}\Lambda)^{-1}$ with Λ stable, $\sup_{z\in\mathbb{D}} \|\Lambda(z)\| < \infty$ s.t. $\det(P + \widetilde{N}\Lambda) \neq 0$. Performance corresponds to

$$\sup_{z\in\mathbb{D}} \|\widetilde{G}_{11} + \widetilde{G}_{12}\Lambda\widetilde{G}_{21}\| \leq 1 \quad (=\text{Model Matching Problem})$$

for $\widetilde{G}_{11} = G_{11} + G_{12}QG_{21}$, $\widetilde{G}_{12} = G_{12}\widetilde{M}$, $\widetilde{G}_{21} = MG_{21}$.

Closed loop system (1D case)

Time domain/state space approach

The closed loop transfer function G_{cl} has system matrix

$$\begin{bmatrix} A_{\mathrm{cl}} & B_{\mathrm{cl}} \\ C_{\mathrm{cl}} & D_{\mathrm{cl}} \end{bmatrix} = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ B_K C_2 & A_K & B_K D_{21} \\ C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{bmatrix}.$$

Closed loop system (1D case)

Time domain/state space approach

The closed loop transfer function G_{cl} has system matrix

$$\begin{bmatrix} A_{\text{cl}} & B_{\text{cl}} \\ C_{\text{cl}} & D_{\text{cl}} \end{bmatrix} = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ B_K C_2 & A_K & B_K D_{21} \\ C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{bmatrix}.$$

TFAE

- $\exists K$ s.t. A_{cl} is stable ($\sigma(A_{\text{cl}}) \subset \mathbb{D}$) ($\Rightarrow G_{\text{cl}}$ stable)
- $\exists F, L$ s.t. $A + BF$ and $A + LC$ stable (operator stabilizable/detectable)
- $\operatorname{Im}[I - zA \;\; B_2] = \mathbb{C}^n$ and $\operatorname{Im}[I - zA^* \;\; C_2^*] = \mathbb{C}^n$, $z \in \overline{\mathbb{D}}$ (Hautus stab/det)
- $\exists X, Y > 0$ s.t. $AXA^* - X - B_2 B_2^* < 0$ and $A^* YA - Y - C_2^* C_2 < 0$

Closed loop system (1D case)

Time domain/state space approach

The closed loop transfer function G_{cl} has system matrix

$$\begin{bmatrix} A_{\text{cl}} & B_{\text{cl}} \\ C_{\text{cl}} & D_{\text{cl}} \end{bmatrix} = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ B_K C_2 & A_K & B_K D_{21} \\ C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{bmatrix}.$$

TFAE

- $\exists K$ s.t. A_{cl} is stable ($\sigma(A_{\text{cl}}) \subset \mathbb{D}$) ($\Rightarrow$ G_{cl} stable)
- $\exists F, L$ s.t. $A + BF$ and $A + LC$ stable (operator stabilizable/detectable)
- $\operatorname{Im}[I - zA \ \ B_2] = \mathbb{C}^n$ and $\operatorname{Im}[I - zA^* \ \ C_2^*] = \mathbb{C}^n$, $z \in \overline{\mathbb{D}}$ (Hautus stab/det)
- $\exists X, Y > 0$ s.t. $AXA^* - X - B_2 B_2^* < 0$ and $A^* YA - Y - C_2^* C_2 < 0$

Performance

- Doyle-Glover-Khargonekar-Francis '89: Coupled Riccati equation solution
- Gahinet-Apkarian '94: Coupled LMI solution. $\exists X, Y > 0$, $\left[\begin{smallmatrix} X & I \\ I & Y \end{smallmatrix}\right] \geq 0$,

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} AYA^* - Y & AYC_1^* & B_1 \\ C_1 YA^* & C_1 YC_1^* - I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0 \quad \text{and} \quad \cdots$$

Givone-Roesser (GR) systems

Now consider the system Σ over d-tuples in $\mathbb{Z}_+$ (with e_i the i-th unit vector):

$$\Sigma := \left\{ \begin{array}{l} \begin{bmatrix} x_1(n+e_1) \\ \vdots \\ x_d(n+e_d) \end{bmatrix} = A \begin{bmatrix} x_1(n) \\ \vdots \\ x_d(n) \end{bmatrix} + B_1 w(n) + B_2 u(n) \\ v(n) = C_1 x(n) + D_{11} w(n) + D_{12} u(n) \\ y(n) = C_2 x(n) + D_{21} w(n) \end{array} \right. \qquad (n \in \mathbb{Z}_+^d),$$

where the state vector decomposes as $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_p}$. Transfer function:

$$G(z) = D + C(I - Z(z)A)^{-1} Z(z) B, \quad Z(z) = \begin{bmatrix} z_1 I_{n_1} & & \\ & \ddots & \\ & & z_d I_{n_d} \end{bmatrix}, \quad z = (z_1, \ldots, z_d) \in \mathbb{C}^d.$$

Givone-Roesser (GR) systems

Now consider the system Σ over d-tuples in $\mathbb{Z}_+$ (with e_i the i-th unit vector):

$$\Sigma := \left\{ \begin{array}{l} \begin{bmatrix} x_1(n+e_1) \\ \vdots \\ x_d(n+e_d) \end{bmatrix} = A \begin{bmatrix} x_1(n) \\ \vdots \\ x_d(n) \end{bmatrix} + B_1 w(n) + B_2 u(n) \\ v(n) = C_1 x(n) + D_{11} w(n) + D_{12} u(n) \\ y(n) = C_2 x(n) + D_{21} w(n) \end{array} \right. \qquad (n \in \mathbb{Z}_+^d),$$

where the state vector decomposes as $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_p}$. Transfer function:

$$G(z) = D + C(I - Z(z)A)^{-1} Z(z) B, \;\; Z(z) = \begin{bmatrix} z_1 I_{n_1} & & \\ & \ddots & \\ & & z_d I_{n_d} \end{bmatrix}, \;\; z = (z_1, \ldots, z_d) \in \mathbb{C}^d.$$

- Hautus stable: $\det(I - Z(z)A) \neq 0$, $z \in \overline{\mathbb{D}}^d \Rightarrow G$ analytic on $\mathbb{D}^d$ (stability)
- Performance: stability & $\|G\|_\infty := \sup_{z \in \mathbb{D}^d} \|G(z)\| \leq 1$.

No stability & Performance: Find a GR controller $K \sim \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ so that the closed loop system $\Sigma_{\rm cl}$ with transfer function $G_{\rm cl}$ has stability and/or performance.

Definition (Scaled stability)

Set $\mathcal{D}_\Sigma = \{\mathrm{diag}\,(X_1, \ldots, X_d) : X_i \in \mathbb{C}^{n_i \times n_i}\}$ (=communtant of $\{Z(z) \colon z \in \mathbb{D}^d\}$). Then A is called *scaled stable* if:

$\exists Q \in \mathcal{D}_\Sigma$, Q invertible s.t. $\|Q^{-1}AQ\| < 1$ ($\Leftrightarrow \exists X \in \mathcal{D}_\Sigma$, $X > 0$ s.t. $AXA^* - X < 0$).

In general: Scaled stability $\neq$ Hautus stability (B.D.O Anderson et. al, 1986)

State space stabilizability and detectability

Definition (Scaled stability)

Set $\mathcal{D}_\Sigma = \{\mathrm{diag}\,(X_1, \ldots, X_d) : \; X_i \in \mathbb{C}^{n_i \times n_i}\}$ (=communtant of $\{Z(z) \colon z \in \mathbb{D}^d\}$). Then A is called *scaled stable* if:

$\exists Q \in \mathcal{D}_\Sigma$, Q invertible s.t. $\|Q^{-1}AQ\| < 1$ ($\Leftrightarrow \exists X \in \mathcal{D}_\Sigma$, $X > 0$ s.t. $AXA^* - X < 0$).

In general: Scaled stability $\neq$ Hautus stability (B.D.O Anderson et. al, 1986)

Definition (state space stabilizability)

The system Σ, or pair $\{A, B_2\}$, is called:

- *Hautus stabilizable* if $\mathrm{Im}\,[I - AZ(z) \;\; B_2] = \mathbb{C}^n$ for all $z \in \overline{\mathbb{D}}^d$;
- *operator stabilizable* if: $\exists F$ such that $A - B_2F$ is Hautus stable
- *scaled stabilizable* if: $\exists F$ such that $A - B_2F$ is scaled stable

Hautus, operator and scaled detectability of Σ, or $\{A, C_2\}$, is defined similarly.

State space stabilizability and detectability

Definition (Scaled stability)

Set $\mathcal{D}_\Sigma = \{\mathrm{diag}\,(X_1, \ldots, X_d) : X_i \in \mathbb{C}^{n_i \times n_i}\}$ (=communtant of $\{Z(z) \colon z \in \mathbb{D}^d\}$). Then A is called *scaled stable* if:

$$\exists Q \in \mathcal{D}_\Sigma, Q \text{ invertible s.t. } \|Q^{-1}AQ\| < 1 \,(\Leftrightarrow \exists X \in \mathcal{D}_\Sigma,\, X > 0 \text{ s.t. } AXA^* - X < 0).$$

In general: Scaled stability $\neq$ Hautus stability (B.D.O Anderson et. al, 1986)

Definition (state space stabilizability)

The system Σ, or pair $\{A, B_2\}$, is called:

- *Hautus stabilizable* if $\mathrm{Im}\,[I - AZ(z)\;\; B_2] = \mathbb{C}^n$ for all $z \in \overline{\mathbb{D}}^d$;
- *operator stabilizable* if: $\exists F$ such that $A - B_2F$ is Hautus stable
- *scaled stabilizable* if: $\exists F$ such that $A - B_2F$ is scaled stable

Hautus, operator and scaled detectability of Σ, or $\{A, C_2\}$, is defined similarly.

Then:

- Σ operator stabilizable & detectable $\Longrightarrow \exists K$ s.t. Σ_{cl} is Hautus stable.
- Σ scaled stabilizable & detectable $\Longleftrightarrow \exists K$ s.t. Σ_{cl} is scaled stable.

Scaled performance, scaled H^∞ problem

The system Σ has *scaled performance* if there exists a $X \in \mathcal{D}_\Sigma$, $X > 0$ so that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I_{\mathcal{W}\oplus\mathcal{U}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* - \begin{bmatrix} X & 0 \\ 0 & I_{\mathcal{W}\oplus\mathcal{U}} \end{bmatrix} < 0.$$

Note: Scaled performance $\Rightarrow$ A scaled stable & $\|G\|_\infty < 1$.
Scaled H^∞ problem: Find a controller K such that Σ_{cl} has scaled performance.

Scaled H^∞ problem

Scaled performance, scaled H^∞ problem

The system Σ has *scaled performance* if there exists a $X \in \mathcal{D}_\Sigma$, $X > 0$ so that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I_{\mathcal{W}\oplus\mathcal{U}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* - \begin{bmatrix} X & 0 \\ 0 & I_{\mathcal{W}\oplus\mathcal{U}} \end{bmatrix} < 0.$$

Note: Scaled performance $\Rightarrow$ A scaled stable & $\|G\|_\infty < 1$.
Scaled H^∞ problem: Find a controller K such that Σ_{cl} has scaled performance.

Theorem (Gahinet-Apkarian '94)

There exists a solution to the scaled H^∞-problem if and only if there exist $X, Y \in \mathcal{D}$, $X, Y > 0$ satisfying LMIs: $\left[\begin{smallmatrix} X & I \\ I & Y \end{smallmatrix}\right] \geq 0$ (coupling condition) and

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} AYA^* - Y & AYC_1^* & B_1 \\ C_1YA^* & C_1YC_1^* - I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0 \quad \text{and} \quad \cdots$$

Here N_c and N_o are full column rank matrices so that $\operatorname{Im} N_c = \operatorname{Ker}\,[B_2^* \;\; D_{12}^*]$ and $\operatorname{Im} N_o = \operatorname{Ker}\,[C_2 \;\; D_{21}]$.

Frequency domain

Systems over rings

In the general setting of System over Rings (Vidyasagar et. al, '80s) the stability and performance problem can be reduced to a Model Matching Problem via coprime factorization (a.o. Quadrat '06).

Frequency domain

Systems over rings

In the general setting of System over Rings (Vidyasagar et. al, '80s) the stability and performance problem can be reduced to a Model Matching Problem via coprime factorization (a.o. Quadrat '06).

What ring to choose?

First choice: Entries of G and K from

$$\mathbb{C}(z)_s = \{p/q\colon\, p, q \in \mathbb{C}(z),\ z \in \mathbb{C}^d,\ p/q \text{ bounded on } \mathbb{D}^d\}.$$

Frequency domain

Systems over rings

In the general setting of System over Rings (Vidyasagar et. al, '80s) the stability and performance problem can be reduced to a Model Matching Problem via coprime factorization (a.o. Quadrat '06).

What ring to choose?

First choice: Entries of G and K from

$$\mathbb{C}(z)_s = \{p/q\colon\ p, q \in \mathbb{C}(z),\ z \in \mathbb{C}^d,\ p/q \text{ bounded on } \mathbb{D}^d\}.$$

Complications:

- For $p/q \in \mathbb{C}(z)_s$ WLOG p, q have no common factor, but thay can still have common zeros; $d \geq 3$: zero varieties of p and q can touch in $\partial\mathbb{D}^d$, while p/q remains bounded on $\mathbb{D}^d$.
- Kharitonov et. al ('99): A arbitrary small perturbation in coefficients of q can give zeros inside $\mathbb{D}^d$.

Systems over rings

In the general setting of System over Rings (Vidyasagar et. al, '80s) the stability and performance problem can be reduced to a Model Matching Problem via coprime factorization (a.o. Quadrat '06).

What ring to choose?

First choice: Entries of G and K from

$$\mathbb{C}(z)_s = \{p/q\colon\, p,q \in \mathbb{C}(z),\, z \in \mathbb{C}^d,\, p/q \text{ bounded on } \mathbb{D}^d\}.$$

Complications:

- For $p/q \in \mathbb{C}(z)_s$ WLOG p, q have no common factor, but thay can still have common zeros; $d \geq 3$: zero varieties of p and q can touch in $\partial\mathbb{D}^d$, while p/q remains bounded on $\mathbb{D}^d$.
- Kharitonov et. al ('99): A arbitrary small perturbation in coefficients of q can give zeros inside $\mathbb{D}^d$.

Lin ('98): Work with

$$\mathbb{C}(z)_{ss} = \{p/q\colon\, p,q \in \mathbb{C}(z) \text{ coprime, } q \text{ no zeros in } \overline{D}^d\}.$$

N.B. A Hautus stable $\Rightarrow$ G has entries in $\mathbb{C}(z)_{ss}$.
G and K entries in $\mathbb{C}(z)_{ss}$: Coprime factorization, reduction to Model Matching

Reduction to interpolation

Model Matching Problem

Given $T_1 \in \mathbb{C}(z)_{ss}^{p\times q}$, $T_2 \in \mathbb{C}(z)_{ss}^{p\times r}$, $T_3 \in \mathbb{C}(z)_{ss}^{s\times q}$, find $\Lambda \in \mathbb{C}(z)_{ss}^{r\times s}$ such that

$$\|T_1 + T_2 \Lambda T_3\|_\infty \leq 1.$$

Model Matching Problem

Given $T_1 \in \mathbb{C}(z)_{ss}^{p\times q}$, $T_2 \in \mathbb{C}(z)_{ss}^{p\times r}$, $T_3 \in \mathbb{C}(z)_{ss}^{s\times q}$, find $\Lambda \in \mathbb{C}(z)_{ss}^{r\times s}$ such that

$$\|T_1 + T_2\Lambda T_3\|_\infty \leq 1.$$

Reduction to interpolation (special case)

Take $p = q = s = 1$ and $T_3 \equiv 1$. Write $T_2 = [T_{2,1} \ \ldots \ T_{2,r}]$. Hence $T_1, T_{2,1}, \ldots, T_{2,r} \in \mathbb{C}(z)_{ss}$. We seek $\Lambda_1, \ldots, \Lambda_r \in \mathbb{C}(z)_{ss}$ so that

$$S = T_1 + T_{2,1}\Lambda_1 + \cdots + T_{2,r}\Lambda_r \text{ satisfies } \|S\|_\infty \leq 1. \tag{1}$$

Model Matching Problem

Given $T_1 \in \mathbb{C}(z)_{ss}^{p\times q}$, $T_2 \in \mathbb{C}(z)_{ss}^{p\times r}$, $T_3 \in \mathbb{C}(z)_{ss}^{s\times q}$, find $\Lambda \in \mathbb{C}(z)_{ss}^{r\times s}$ such that

$$\|T_1 + T_2 \Lambda T_3\|_\infty \leq 1.$$

Reduction to interpolation (special case)

Take $p = q = s = 1$ and $T_3 \equiv 1$. Write $T_2 = [T_{2,1} \ \ldots \ T_{2,r}]$. Hence $T_1, T_{2,1}, \ldots, T_{2,r} \in \mathbb{C}(z)_{ss}$. We seek $\Lambda_1, \ldots, \Lambda_r \in \mathbb{C}(z)_{ss}$ so that

$$S = T_1 + T_{2,1}\Lambda_1 + \cdots + T_{2,r}\Lambda_r \text{ satisfies } \|S\|_\infty \leq 1. \tag{1}$$

Assume the zero varieties of $T_{2,1}, \ldots, T_{2,r}$ intersect within $\overline{\mathbb{D}}^d$ only in finitely many points $z^(1), \ldots, z^{(k)} \in \overline{\mathbb{D}}^d$. Set $w_j = T_1(z^j)$, $j = 1, \ldots, k$. Then $S \in \mathbb{C}(z)_{ss}$ is of the form (1) if and only if

$$S(z^{(j)}) = w_j, \ j = 1, \ldots, k \quad \text{and} \quad \|S\|_\infty \leq 1.$$

Model Matching Problem

Given $T_1 \in \mathbb{C}(z)_{ss}^{p\times q}$, $T_2 \in \mathbb{C}(z)_{ss}^{p\times r}$, $T_3 \in \mathbb{C}(z)_{ss}^{s\times q}$, find $\Lambda \in \mathbb{C}(z)_{ss}^{r\times s}$ such that

$$\|T_1 + T_2 \Lambda T_3\|_\infty \leq 1.$$

Reduction to interpolation (special case)

Take $p = q = s = 1$ and $T_3 \equiv 1$. Write $T_2 = [T_{2,1} \ \dots \ T_{2,r}]$. Hence $T_1, T_{2,1}, \dots, T_{2,r} \in \mathbb{C}(z)_{ss}$. We seek $\Lambda_1, \dots, \Lambda_r \in \mathbb{C}(z)_{ss}$ so that

$$S = T_1 + T_{2,1}\Lambda_1 + \cdots + T_{2,r}\Lambda_r \text{ satisfies } \|S\|_\infty \leq 1. \tag{1}$$

Assume the zero varieties of $T_{2,1}, \dots, T_{2,r}$ intersect within $\overline{\mathbb{D}}^d$ only in finitely many points $z^(1), \dots, z^{(k)} \in \overline{\mathbb{D}}^d$. Set $w_j = T_1(z^j)$, $j = 1, \dots, k$. Then $S \in \mathbb{C}(z)_{ss}$ is of the form (1) if and only if

$$S(z^{(j)}) = w_j, \ j = 1, \dots, k \quad \text{and} \quad \|S\|_\infty \leq 1.$$

Issues:

- Not finitely many points: interpolation along a variety
- Characterization of functions in $\mathbb{C}(z)_{ss}$?

What functions are transfer functions of stable GR systems?

Extend to analytic functions on $\mathbb{D}^d$ and possibly infinite dimensional systems with contractive system matrices: $\| \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \| \leq 1$.

Theorem (Agler '00)

A matrix function G on $\mathbb{D}^d$ is the transfer function of a contractive GR system, which implies $I - Z(z)A$ invertible for $z \in \mathbb{D}^d$ and $\|G\|_\infty \leq 1$, if and only if G is analytic on $\mathbb{D}^d$, say $G(z) = \sum_{n \in \mathbb{Z}_+^d} G_n z^n$, and for any d commuting strict contractions $X_1, \ldots, X_d$ on ℓ^2 we have $\|G(X_1, \ldots, X_d)\| \leq 1$, where

$$G(X_1, \ldots, X_d) = \sum_{n \in \mathbb{Z}_+^d} G_n \otimes X^n, \quad \text{with} \quad X^n = X_1^{n_1} \cdots X_d^{n_d}, \text{ if } n = (n_1, \ldots, n_d).$$

The class of such function is called the Schur-Agler class.

What functions are transfer functions of stable GR systems?

Extend to analytic functions on $\mathbb{D}^d$ and possibly infinite dimensional systems with contractive system matrices: $\| \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \| \leq 1$.

Theorem (Agler '00)

A matrix function G on $\mathbb{D}^d$ is the transfer function of a contractive GR system, which implies $I - Z(z)A$ invertible for $z \in \mathbb{D}^d$ and $\|G\|_\infty \leq 1$, if and only if G is analytic on $\mathbb{D}^d$, say $G(z) = \sum_{n \in \mathbb{Z}_+^d} G_n z^n$, and for any d commuting strict contractions $X_1, \ldots, X_d$ on ℓ^2 we have $\|G(X_1, \ldots, X_d)\| \leq 1$, where

$$G(X_1, \ldots, X_d) = \sum_{n \in \mathbb{Z}_+^d} G_n \otimes X^n, \quad \text{with} \quad X^n = X_1^{n_1} \cdots X_d^{n_d}, \text{ if } n = (n_1, \ldots, n_d).$$

The class of such function is called the Schur-Agler class.

Agler's interpolation theorem ('00)

Given $z^{(1)}, \ldots, z^{(k)} \in \mathbb{D}^d$ and $w_1, \ldots, w_k \in \mathbb{C}$, there exists a Schur-Agler function S so that

$$S(z^{(j)}) = w_j \quad \text{for} \quad j = 1, \ldots, k$$

if and only if there exists $k \times k$ matrices $P^{(1)}, \ldots, P^{(d)} \geq 0$ so that

$$1 - w_i \overline{w_j} = \sum_{l=1}^{d} (1 - z_l^{(i)} \overline{z}_l^{(j)}) P_{i,j}^{(l)}, \quad i, j = 1, \ldots, k.$$

More recent developments

Issues with Givone-Roesser systems

Desirable stability notion $\neq$ computable stability notion; no minimality, controllability, observability, Kalman decomposition.

More recent developments

Issues with Givone-Roesser systems

Desirable stability notion $\neq$ computable stability notion; no minimality, controllability, observability, Kalman decomposition.

Mathematical solution (2005–)

Evaluate in *d*-tuples of (possibly) noncommutative operators.
Different system: evolution along free semigroup in d letters, rather than $\mathbb{Z}_+^d$.
Then minimality & Kalman decomposition extist; Hautus, operator & scaled stability coincide, etc.

More recent developments

Issues with Givone-Roesser systems

Desirable stability notion $\neq$ computable stability notion; no minimality, controllability, observability, Kalman decomposition.

Mathematical solution (2005–)

Evaluate in d-tuples of (possibly) noncommutative operators.
Different system: evolution along free semigroup in d letters, rather than $\mathbb{Z}_+^d$.
Then minimality & Kalman decomposition extist; Hautus, operator & scaled stability coincide, etc.

Appeared earlier, and is still used, in engineering literature

- Lu-Zhou-Doyle '91: Notions of stability coincide
- Paganini '96, Packard '94: Gain scheduling with or without time-varying uncertainty parameters
- Poola-Tikku '95: slowly time-varying systems
- Köroğlu-Scherer '07: nonsquare blocks, bounds on time-variation
- Scherer-Köse '12: frequency-dependent D-scaling for gain-scheduled feedback configuration.

THANK YOU FOR YOUR ATTENTION

For more details and proper references see:

- J.A. Ball and S. ter Horst, Robust control, multidimensional systems and multivariable Nevanlinna-Pick interpolation, *Oper. Theory Adv. Appl.* **203** (2010), 13–88.