Error estimates for the Galerkin finite element approximation for a linear second order hyperbolic equation with modal damping

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SANUM 2016
22 March 2016

Outline

General linear vibration model

Let X, W and V denote Hilbert spaces such that $V \subset W \subset X$

Space	Inner product	Norm
X	$(\cdot,\cdot)_X$	$\|\cdot\|_X$
Inertia space W	$c(\cdot,\cdot)$	$\|\cdot\|_W$
Energy space V	$b(\cdot,\cdot)$	$\|\cdot\|_V$

J is an open interval containing zero, or of the form $[0,\tau)$ or $[0,\infty)$.

Problem G
Given a function $f : J \to X$, find a function $u \in C\big(J; V\big)$ such that u' is continuous at 0, *and for each $t \in J$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$, and*

$$c\big(u''(t), v\big) + a\big(u'(t), v\big) + b\big(u(t), v\big) = \big(f(t), v\big)_X \quad \textit{for each } v \in V, \tag{1}$$

while $u(0) = u_0$ and $u'(0) = u_1$

General linear vibration model

Assumptions
We assume that the following additional properties hold

- **E1** V is dense in W and W is dense in X
- **E2** There exists a constant C_b such that $\|v\|_W \leq C_b\|v\|_V$ for each $v \in V$
- **E3** There exists a constant C_c such that $\|v\|_X \leq C_c\|v\|_W$ for each $v \in W$
- **E4** The bilinear form a is nonnegative, symmetric and bounded on V

Viscous type damping

$$|a(u, v)| \leq K_a \|u\|_W \|v\|_W$$

M. Basson and N. F. J. van Rensburg (2013) Galerkin finite element approximation of general linear second order hyperbolic equations, Numerical Functional Analysis and Optimization, 34:9, 976 - 1000
DOI: 10.1080/01630563.2013.807286

Modal damping

$$a(u,v) = \mu b(u,v) + \eta c(u,v); \mu \geq 0, \eta \geq 0$$

Viscous damping (air damping, external damping)

Material damping (strain rate damping, Kelvin-Voigt damping, internal damping)

Example: Euler-Bernoulli beam model with viscous damping and internal damping

Hyperbolic heat conduction

Bounded domain $\Omega \subset \mathbb{R}^3$
Conservation of heat energy:

$$\rho c_p \partial_t T = -\nabla \cdot q + f$$

Density ρ, the specific heat c_p, temperature T
Heat source term f, heat flux q

Fourier's law:	Cattaneo-Vernotte law:	Generalised dual-phase-lag:
$q = -k\nabla T$	$q(r, t + \tau_q) = -k\nabla T(r, t)$ $q + \tau_q \partial_t q = -k\nabla T$	$q(r, t + \tau_q) = -A\nabla T(r, t + \tau_T)$ $q + \tau_q \partial_t q = -A\nabla T - \tau_T A \partial_t \nabla T$
Heat equation:	Hyperbolic heat equation:	Generalised dual-phase-lag model:
$\partial_t T = c^2 \nabla^2 T$	$\tau_q \partial_t^2 T + \partial_t T = c^2 \nabla^2 T$	$\gamma_2 \partial_t^2 T + \gamma_1 \partial_t T - \tau_T \nabla \cdot (A\nabla(\partial_t T))$ $= \nabla \cdot (A\nabla T)$

Thermal conductivity k, and $c^2 = \frac{k}{\rho c_p}$
Time delay τ_q in heat flux, and time delay τ_T for temperature gradient
$\gamma_1 = \rho c_p$, and $\gamma_2 = \tau_q \rho c_p$

Galerkin approximation

S^h is a finite dimensional subspace of V

Problem G^h
Given a function $f : J \to X$, find a function $u_h \in C^2(J)$ such that u_h' is continuous at 0 and for each $t \in J$, $u_h(t) \in S^h$ and

$$c\big(u_h''(t), v\big) + a\big(u_h'(t), v\big) + b\big(u_h(t), v\big) = \big(f(t), v\big)_X \text{ for each } v \in S^h, \tag{2}$$

while $u_h(0) = u_0^h$ and $u_h'(0) = u_1^h$

The initial values u_0^h and u_1^h are elements of S^h as close as possible to u_0 and u_1

Semi-discrete approximation

A projection is used to find an estimate for the *discretization error*

$$e_h(t) = u(t) - u_h(t)$$

The projection operator P is defined by

$$b(u - Pu, v) = 0 \quad \text{for each } v \in S^h$$

The idea of the projection method is to split the error $e_h(t)$ into two parts

$$e_h(t) = e_p(t) + e(t) = (u(t) - Pu(t)) + (Pu(t) - u_h(t))$$

Fundamental estimate

Lemma

If the solution u of Problem G satisfies $Pu \in C^2(J)$, then for any $t \in J$,

$$\|e(t)\|_V + \|e'(t)\|_W \leq \sqrt{2}\left(\|e(0)\|_V + \|e'(0)\|_W + \int_0^t (\|e_p''\|_W + \eta\|e_p'\|_W)\right)$$

The proof is based on the brief outline given in Strange and Fix, *An Analysis of the Finite Element Method* (1973) for the undamped wave equation.
The energy expression $E(t)$ for $e(t)$ forms the central concept in the proof

$$E(t) = \frac{1}{2}c(e'(t), e'(t)) + \frac{1}{2}b(e(t), e(t)) = \frac{1}{2}\|e'(t)\|_W^2 + \frac{1}{2}\|e(t)\|_V^2$$

Projection error

There exists a subspace $H(V,k)$ of V, and positive constants C and α (depending on V and k) such that for $u \in H(V,k)$,

$$\|u - Pu\|_V \leq Ch^{\alpha}\|u\|_{H(V,k)},$$

where $\|\cdot\|_{H(V,k)}$ is a norm or semi-norm associated with $H(V,k)$
k is a positive integer determined by the regularity of the solution u

Galerkin approximation: semi-discrete approximation

$u_h(t) = \sum_{j=1}^{n} u_j(t)\phi_j$ if $\{\phi_j\}$ forms a basis for S^h.

Semi-discrete approximation

$$M\bar{u}''(t) + C\bar{u}'(t) + K\bar{u}(t) = \bar{F}(t)$$

$\bar{u}(t) = [u_1(t)\ u_2(t)\ \ldots\ u_n(t)]^t$

Approximation of $\bar{u}(t_k)$ is $\bar{u}_k$

Fully discrete approximation

$$(\delta t)^{-2}M[\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}] + (2\delta t)^{-1}C[\bar{u}_{k+1} - \bar{u}_{k-1}] + K\bar{u}_k = \bar{F}(t_k)$$

Galerkin approximation: fully discrete approximation

If $\bar{u}_k = (u_k^1, u_k^2, \dots, u_k^n)$, then $u_k^h = \sum_{j=1}^n u_k^j \phi_j$ is the approximation for $u_h(t_k)$.

Problem $G^h D$
Assume that ρ_0 and ρ_1 are positive numbers such that $\rho_0 + 2\rho_1 = 1$. Find a sequence $\{u_k^h\} \subset S^h$ such that for each $k = 1,\ 2, \dots,\ N-1$,

$$\begin{aligned} & c(\delta t^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a((2\delta t)^{-1}[u_{k+1}^h - u_{k-1}^h], v) \qquad (3) \\ + \quad & b(\rho_1 u_{k+1}^h + \rho_0 u_k^h + \rho_1 u_{k-1}^h, v) = (\rho_1 f(t_{k+1}) + \rho_0 f(t_k) + \rho_1 f(t_{k-1}), v)_X, \end{aligned}$$

for each $v \in S^h$ while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = (2\delta t)v^h$.

Galerkin approximation: fully discrete approximation

Error estimate

$$\begin{aligned} \|u(t_k) - u_k^h\|_W &\leq \|u(t_k) - u_h(t_k)\|_W + \|u_h(t_k) - u_k^h\|_W \\ &\leq K_2 h^\alpha + K_1 \delta t^2. \end{aligned}$$