# How sound is our acoustics?

Niko Sauer

Centre for the Advancement of Scholarship
University of Pretoria

# Layout

No soft analysis (spaces).

No generalizations.

Ideal isentropic gas.

Eulerian description: (introduced by d'Alembert)

$$\left.\begin{array}{l}\rho[\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}] + \nabla p = 0;\\ \rho_t + \nabla\cdot(\rho\mathbf{v}) = 0.\end{array}\right\}$$

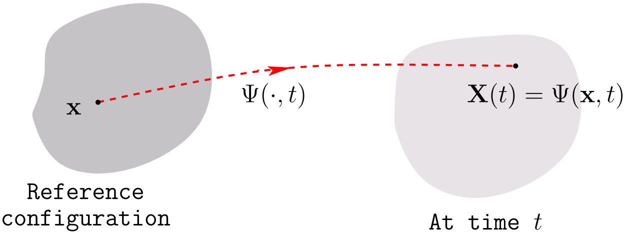
$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t); \quad p = p(\mathbf{x}, t); \quad \rho(\mathbf{x}, t)$$

observed at a fixed point $\mathbf{x} = (x, y, z)$ in space at time $t$.

Need a *thermodynamic relation* between $\rho$ and $p$.

Lagrangian description (introduced by Euler).

Follow particle **x** in a *reference configuration*.




The moving body of gas consists of the same material.

$$\rho(\mathbf{x}) = J(\mathbf{x}, t)\sigma(\Psi(\mathbf{x}, t), t).$$

$\rho(\mathbf{x}) =$ mass density in the reference configuration.
$J(\mathbf{x}, t) = \partial\mathbf{X}/\partial\mathbf{x} =$ Jacobian of $\Psi$.
$\sigma(\mathbf{X}, t) =$ mass density at time $t$.

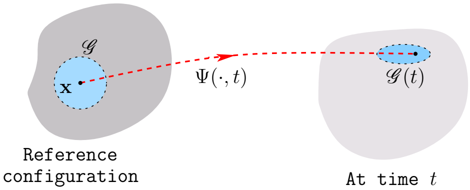
This means that $J(\mathbf{x}, t)$ must be positive.

Combined with conservation of mass:

$$\rho(\mathbf{x})\mathbf{v}_t(\mathbf{x}, t) + J(\mathbf{x}, t)[\nabla_x \Psi(\mathbf{x}, t)]^{-T} \nabla_x p(\mathbf{x}, t) = 0.$$

At time $t$:

$\mathbf{v}(\mathbf{x}, t) =$ velocity of $\mathbf{x}$.
$p(\mathbf{x}, t) =$ pressure experienced by $\mathbf{x}$.




Compression:

$$r(\mathbf{x}, t) := \lim_{\substack{|\mathscr{G}| \to 0 \\ \mathbf{x} \in \mathscr{G}}} \frac{|\mathscr{G}(t)| - |\mathscr{G}|}{|\mathscr{G}|} = J(\mathbf{x}, t) - 1.$$

Acoustic assumption (from thermostatics):

$$[1 + \tfrac{p(\mathbf{x},t) - p_0}{\rho c^2}][1 + r(\mathbf{x}, t)] = 1.$$

$c =$ Thermostatic sound speed.

The acoustic assumption (and Euler's formula) leads to:

$$\mathbf{v}_t(\mathbf{x},t) + \left[\frac{c^2}{\rho c^2 + p(\mathbf{x},t) - p_0}\right][\nabla_x \Psi]^{-T} \nabla_x p(\mathbf{x},t) = 0,$$

$$p_t(\mathbf{x},t) + (\rho c^2 + p - p_0)[\nabla_x \Psi]^{-T} : \nabla_x \mathbf{v}(x,t) = 0,$$

and the constraint:

$$1 + \frac{p(\mathbf{x},t) - p_0}{\rho c^2} > 0.$$

Assume that $\rho$ is constant. Let $L$ be a chosen unit of length and let $T = L/c$ be the new unit of time. We scale to dimensionless quantities in the following way:

$$\mathbf{x} \longrightarrow \mathbf{x}/L; \quad t \longrightarrow t/T;$$
$$\mathbf{v} \longrightarrow \mathbf{v}/c; \quad p \longrightarrow (p - p_0)/\rho c^2.$$

The equations now become:

$$\mathbf{v}_t + [1+p]^{-1}[\nabla_x \Psi]^{-T} \nabla_x p = 0;$$
$$p_t + [1+p][\nabla_x \Psi]^{-T} : \nabla_x \mathbf{v} = 0.$$

Also:

$1 + p > 0;$ Constraint.

$(1+p)(1+r) = 1;$ Acoustic assumption (Boyle-Mariott).

$$\Psi(\mathbf{x}, t) = (\psi(x, t), y, z).$$

The equations simplify to

$$\left.\begin{aligned} v_t(x, t) + p_x(x, t) &= 0; \\ p_t(x, t) + [1 + p(x, t)]^2 v_x(x, t) &= 0. \end{aligned}\right\}$$

$\mathbf{v} = (v, 0, 0)$; $-\infty < x < \infty$; $t > 0$.

The constraint:

$$1 + p(x, t) > 0.$$

Ensures that the system is hyperbolic.

Eliminate $v$ by (carefree) differentiation. The result:

$$p_{tt} - \frac{2}{1+p}[p_t]^2 - [1+p]^2 p_{xx} = 0.$$

Let's get away from here!

Substitution

$$1 + p(x,t) = \exp\{q(x,t)\} > 0.$$

Equations:

$$\left.\begin{aligned} v_t(x,t) + \exp\{q(x,t)\}q_x(x,t) = 0; \\ q_t(x,t) + \exp\{q(x,t)\}v_x(x,t) = 0. \end{aligned}\right\}$$

Note:

- The system is symmetric hyperbolic.
- The roles of $v$ and $q$ cannot be interchanged.

More substitutions:

$$u_1(x,t) = \tfrac{1}{2}[v+q]; \quad u_2(x,t) = \tfrac{1}{2}[v-q].$$

$$v = u_1 + u_2; \quad q = u_1 - u_2.$$

New equations:

$$\left.\begin{aligned} u_{1,t} + \exp\{q\} u_{1,x} &= 0; \\ u_{2,t} + \exp\{q\} u_{2,x} &= 0. \end{aligned}\right\}$$

Characteristics. $\mathrm{C}_1 : x = X_1(t); \quad \mathrm{C}_2 : x = X_2(t).$

$$X_1'(t) = \exp\{q(X_1(t), t)\};$$

$$X_2'(t) = -\exp\{q(X_2(t), t)\}.$$

$u_1$ is constant along $\mathrm{C}_1$;
$u_2$ is constant along $\mathrm{C}_2$.

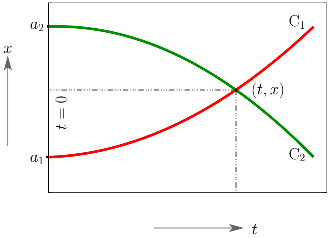
At time $t = 0$:

$$\left.\begin{aligned} v(x,t)\big|_{t=0} &= v_0(x), \\ p(x,t)\big|_{t=0} &= p_0(x). \end{aligned}\right\}$$

$$1 + p_0(x) = \exp\{q_0(x)\} > 0.$$

$$\left.\begin{aligned} u_{10} = u_1\big|_{t=0} &= \tfrac{1}{2}[v_0 + q_0], \\ u_{20} = u_2\big|_{t=0} &= \tfrac{1}{2}[v_0 - q_0]. \end{aligned}\right\}$$

A picture:




$$a_1 = a_1(t, x) = X_1(0); \quad a_2 = a_2(t, x) = X_2(0)$$

are called *inverse characteristics.*

$$u_1(x, t) = u_{10}(a_1(t, x));$$
$$u_2(x, t) = u_{20}(a_2(t, x)).$$

$$X_1'(t) = \exp\{q(X_1(t), t)\}; \quad X_1(0) = a_1(t,x); \quad X_1(t) = x.$$

$$\begin{aligned} x - a_1(t,x) = \int_0^t X_1'(s)\,ds &= \int_0^t \exp\{q(X_1(s), s)\}\,ds \\ &= \quad \ldots \quad \ldots \\ &= \exp\{u_{10}(a_1(t,x))\} \int_0^t \exp\{-u_2(X_1(s), s)\}\,ds. \end{aligned}$$

Similarly,

$$a_2(t,x) - x = \exp\{-u_{20}(a_2(t,x))\} \int_0^t \exp\{u_1(X_2(s), s)\}\,ds.$$

Differentiate $\partial_t$ and work hard. For fixed $x$:

$$\left.\begin{aligned}[1+(x-a_1)u_{10}'(a_1)]a_{1,t} &= -\exp\{u_{10}(a_1)-u_{20}(a_2)\};\\ [1+(a_2-x)u_{20}'(a_2)]a_{2,t} &= \exp\{u_{10}(a_1)-u_{20}(a_2)\}.\end{aligned}\right\}$$

Let $b_1 = x - a_1 > 0;\ b_2 = a_2 - x > 0.$ Then,

$$[1+b_1 u_{10}'(x-b_1)]b_{1,t} = [1+b_2 u_{20}'(x+b_2)]b_{2,t}.$$

Integrate (with appropriate substitutions).

$$\int_0^{b_1}[1+\sigma u_{10}'(x-\sigma)]\,d\sigma = \int_0^{b_2}[1+\sigma u_{20}'(x+\sigma)]\,d\sigma.$$

To calculate a phase portrait. A possible algoritm:

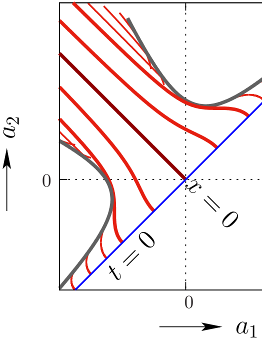
1. Fix $x$.
2. Choose $b_1$.
3. Solve for $b_2$ from

$$\int_0^{b_1} [1 + \sigma u'_{10}(x - \sigma)]\, d\sigma = \int_0^{b_2} [1 + \sigma u'_{20}(x + \sigma)]\, d\sigma.$$

4. Calculate: $a_1 = x - b_1$; $a_2 = x + b_2$.
5. Change $b_1$. Go to 3.

   ...

6. Change $x$. Go to 2.




**Important:** Unrestrained trajectories.

They define a useful new (warped) time.

Some areas are never visited.

$$\int_0^{b_1} [1 + \sigma u'_{10}(x - \sigma)]\, d\sigma = \int_0^{b_2} [1 + \sigma u'_{20}(x + \sigma)]\, d\sigma.$$

The trajectory will be unrestrained if for some $x_0$ the equations

$$1 + \sigma u'_{10}(x_0 - \sigma)] = 0;$$
$$1 + \sigma u'_{20}(x_0 + \sigma) = 0,$$

have no positive solutions.

We may take $x_0 = 0$.

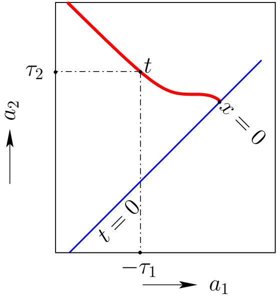
Define new times:

$$\tau_1 = b_1 = -a_1,$$
$$\tau_2 = b_2 = a_2.$$

$$\int_0^{\tau_1} [1 + \sigma u_{10}'(-\sigma)]\, d\sigma = \int_0^{\tau_2} [1 + \sigma u_{20}'(\sigma)]\, d\sigma.$$

Properties:

- $\tau_1 = \tau_1(t)$; $\tau_2 = \tau_2(t)$.
- $\tau_1 \to \tau_2$ is one-one.
- $\tau_1(t) \to \infty$; $\tau_2(t) \to \infty$ as $t \to \infty$.



## Solution curves

Aim: calculation of the curves $x \to v(x,t)$, $x \to p(x,t)$ for fixed $t > 0$.

Recall the representations:

$$x - a_1(t,x) = \exp\{u_{10}(a_1(t,x))\} \int_0^t \exp\{-u_2(X_1(s),s)\}\, ds;$$

$$a_2(t,x) - x = \exp\{-u_{20}(a_2(t,x)\} \int_0^t \exp\{u_1(X_2(s),s)\}\, ds.$$

This time differentiate $\partial_x$ to obtain a second system of ODE's:

$$\left.\begin{array}{r}[1 + (x - a_1)u'_{10}(a_1)]a_{1,x} = 1; \\ [1 + (a_2 - x)u'_{20}(a_2)]a_{2,x} = 1.\end{array}\right\}$$

Singularities can occur when terms in brackets are zero.

Imagine $x$ as a function of $a_1$ or $a_2$.

Formally write (for fixed $t$):

$$\frac{dx_1}{da_1} = 1 + (x_1 - a_1)u'_{10}(a_1);$$
$$\frac{dx_2}{da_2} = 1 - (a_2 - x_2)u'_{20}(a_2).$$

Point conditions: $a_1(t,0) = -\tau_1,\ a_2(t,0) = \tau_2$.

$$x_1(-\tau_1) = 0;$$
$$x_2(\tau_2) = 0.$$

Singularities at local extrema of $x_1,\ x_2$.

Generalized pressures:

$$P_1(a) := \exp\{u_{10}(a)\};$$
$$P_2(a) := \exp\{-u_{20}(a)\}.$$

Explicit solutions:

$$x_1(a_1;\tau_1) = a_1 + \tfrac{\tau_1}{P_1(-\tau_1)} P_1(a_1);$$
$$x_2(a_2;\tau_2) = a_2 - \tfrac{\tau_2}{P_2(\tau_2)} P_2(a_2).$$

1. Choose $\tau_1$.
2. Calculate $\tau_2$ from
   $$\int_0^{\tau_1}[1+\sigma u_{10}'(-\sigma)]\,d\sigma = \int_0^{\tau_2}[1+\sigma u_{20}'(\sigma)]\,d\sigma.$$
3. Choose $x$.
4. Calculate $a_1$ from $x_1(a_1;\tau_1)=x$ and $a_2$ from $x_2(a_2;\tau_2)=x$.
5. Now,
   $u_1(x,t)=u_{10}(a_1);\quad u_2(x,t)=u_{20}(a_2);$
   $v(x,t)=u_1+u_2;\quad q(x,t)=u_1-u_2;\quad 1+p(x,t)=\exp\{q\}.$
6. Change $x$. Go to 4.

   ...

7. Change $\tau_1$. Go to 2.

# Oops

We forgot about singularities!

Consider the curve on which $a_1$ may be singular. That is where $1 + (x - a_1)u'_{10}(a_1) = 0$. It is denoted by $\Sigma_1$ (after the river *Styx*) and given by

$$x = S(a_1) := a_1 - \frac{1}{u'_{10}(a_1)}\,.$$

Where the curve $x_1(a_1; \tau_1)$ crosses Styx, $x'_1(a_1; \tau_1) = 0$ and

$$\frac{P_1(-\tau_1)}{\tau_1} = -P'_1(a_1).$$

If $u'_{10}(a_1) > 0$, $P'_1(a_1) = u'_{10}(a_1)P_1(a_1) > 0$ and the crossing will not happen.

Assume:

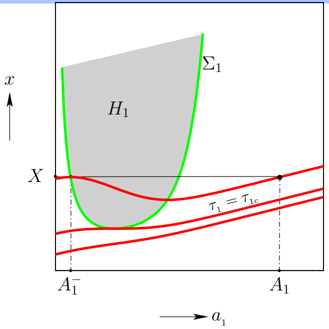
- $P_1'(a_1) < 0$ on some interval $I = (A_1^\dagger, \infty)$.
- $S_1(a_1) \to \infty$ when $a_1 \to A_1^\dagger$.
- The function $S(a_1)$ is strictly convex on $I$.
- $\tau_1 P_1'(-\tau_1) + P_1(-\tau_1) > 0$.

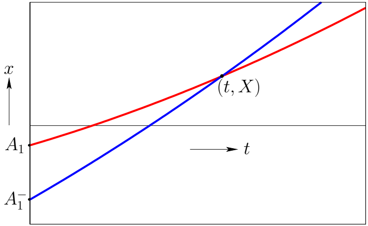
Let $H_1$ (for *Hades*) be the region above $\Sigma_1$ (*Styx*).

1. $S(a_1) \to \infty$ as $a_1 \to \infty$. Hades is quite large — and full of musical amateurs.

2. The curve $x = x_1(a_1; \tau_1)$
   - hits $\Sigma_1$ only where $x_1'(a_1; \tau_1) = 0$,
   - is inside $H_1$ if and only if $x_1'(a_1; \tau_1) < 0$,
   - is elsewhere if and only if $x_1'(a_1; \tau_1) > 0$.

3. There is unique $\tau_{1c} > 0$ such that the curve $x = x_1(a_1; \tau_1)$
   - is below $\Sigma_1$ if $0 \leq \tau_1 < \tau_{1c}$;
   - touches $\Sigma_1$ if $\tau_1 = \tau_{1c}$;
   - crosses $H_1$ if $\tau_1 > \tau_{1c}$.




$u_1$ is discontinuous.

Note: $X = x_1(A_1^-; \tau_1) = x_1(A_1; \tau_1)$; $\quad A_1^- < A_1$.

Thus, if $\tau_1 = \tau_1(t)$,

$a_1(X, t) = A_1^-$ and $a_1(X, t) = A_1$.

$a_1$ is multi-valued.

Assume that $P_2'(a_2) > 0$ for $a_2 \in I$.

- Then $a_2(x,t)$ is single-valued.
- Recall that $u_1(x,t) = u_{10}(a_1(t,x))$ and $u_2(x,t) = u_{20}(a_2(t,x))$.
- There is a *jump discontinuity* in $u_1$ at $x = X$.
- $v(x,t) = u_1(x,t) + u_2(x,t)$ and $q(x,t) = u_1(x,t) - u_2(x,t)$.
- Thus $v(x,t)$ and $p(x,t) = \exp\{q(x,t)\} - 1$ are discontinuous at $x = X$ and some $t > 0$.

This is a *shock phenomenon*.

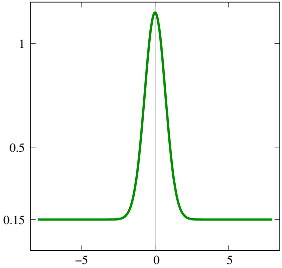
Let us model an explosion in the following way:

$$v_0(x) = 0$$

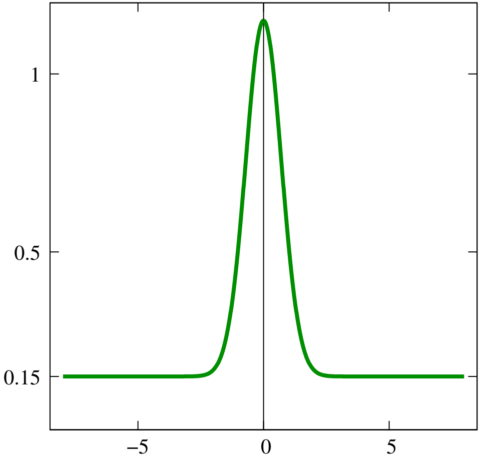
and

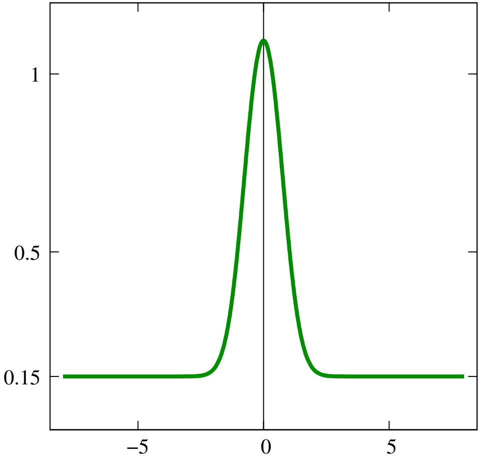
$$1 + p_0(x) = \exp\{-x^2\} + m; \;\; m \geq 0.$$

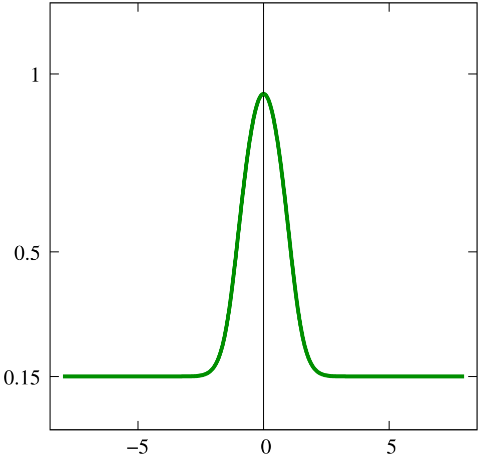
Here is a picture:

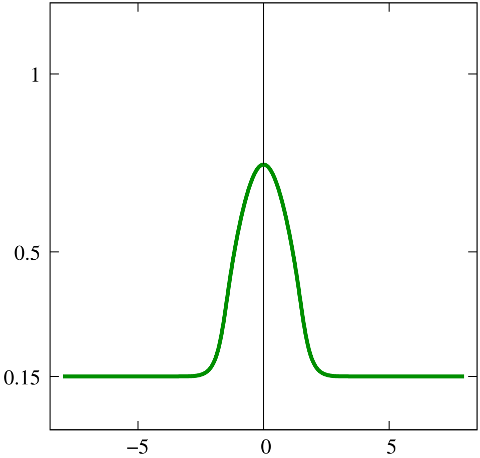


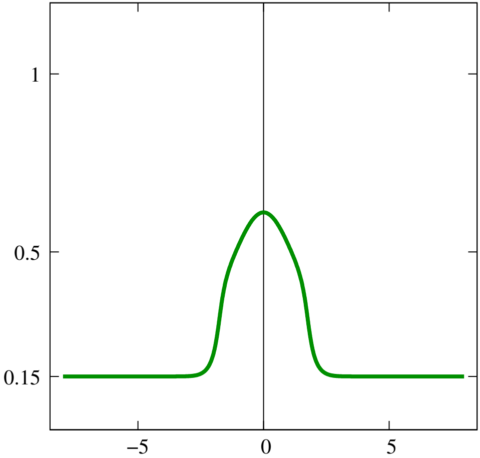

Watch this space.


1
0.5
0.15
−5
0
5

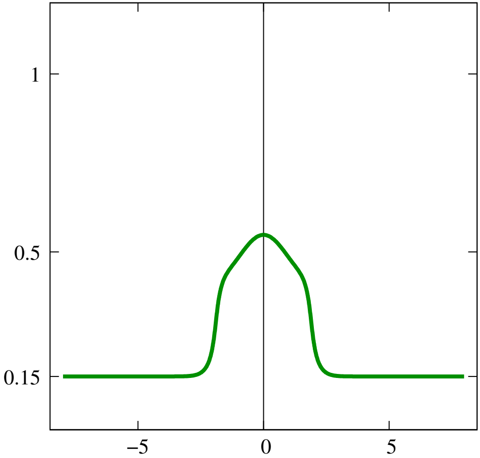

1
0.5
0.15
–5
0
5


1
0.5
0.15
−5
0
5

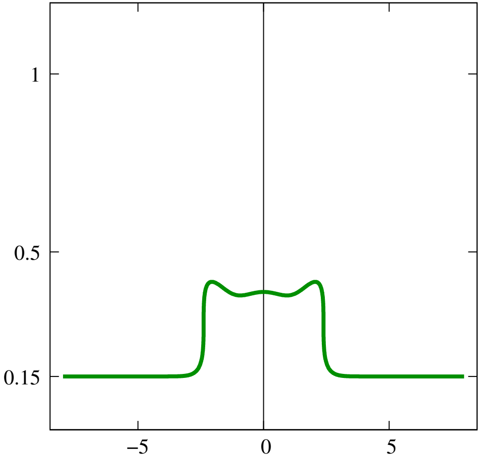

1
0.5
0.15
–5
0
5



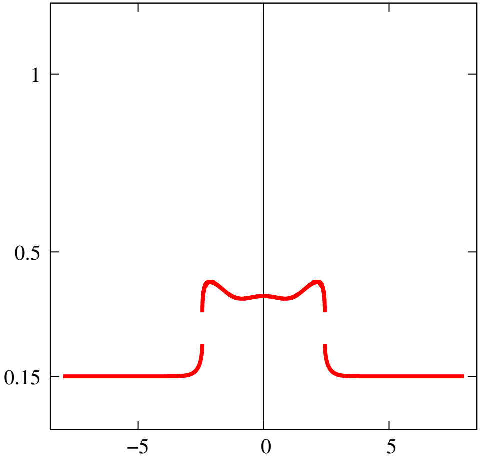
1
0.5
0.15
−5
0
5

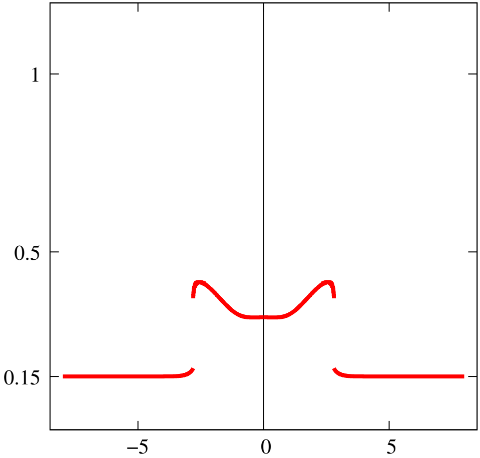


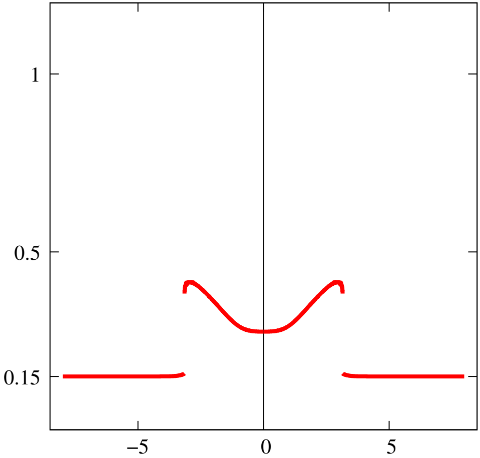
1
0.5
0.15
–5
0
5

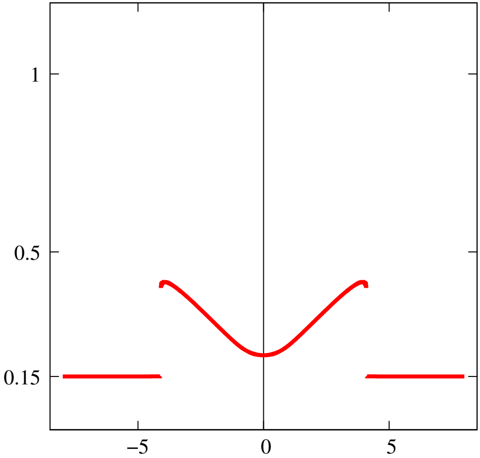

1
0.5
0.15
–5
0
5

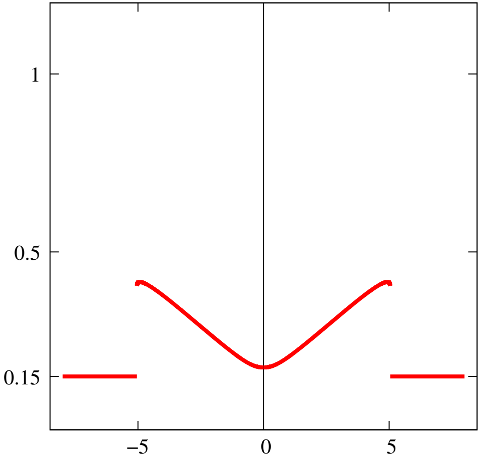
# Pressure

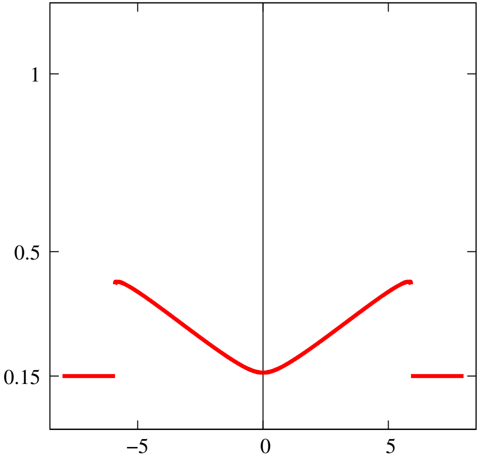


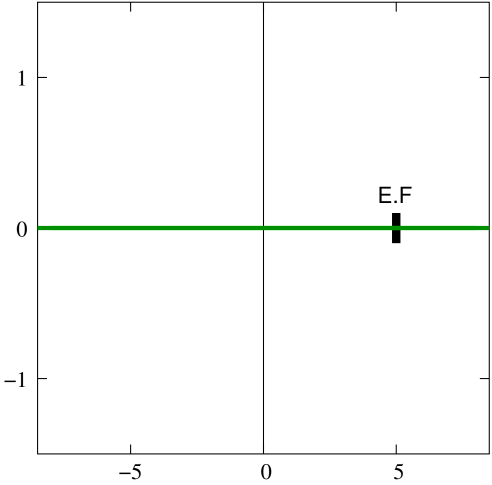

1
0.5
0.15
–5
0
5


1
0.5
0.15
–5
0
5

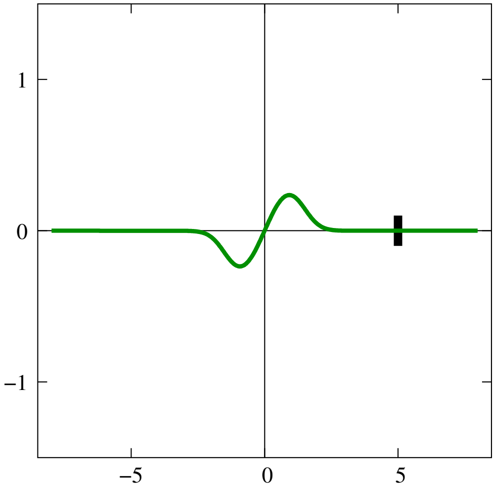

1
0.5
0.15
−5
0
5

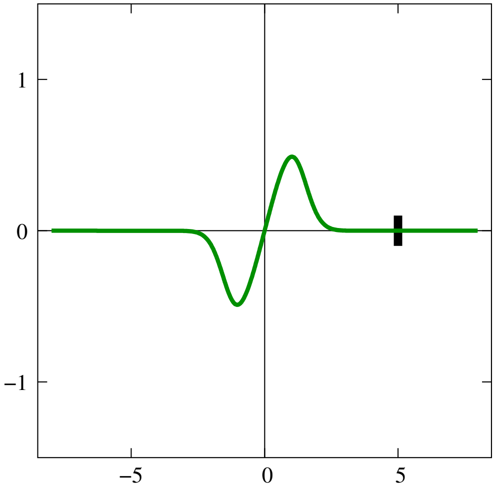

1
0.5
0.15
−5
0
5

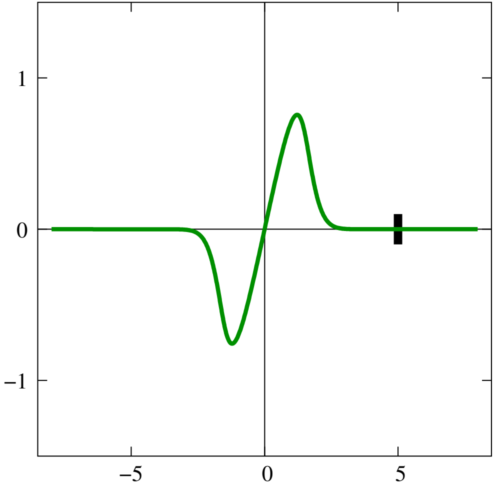

1
0.5
0.15
–5
0
5

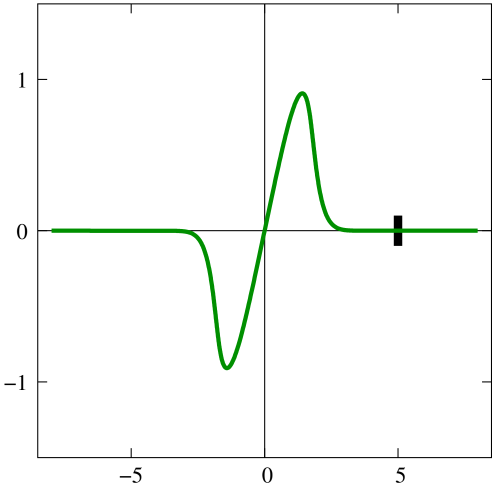


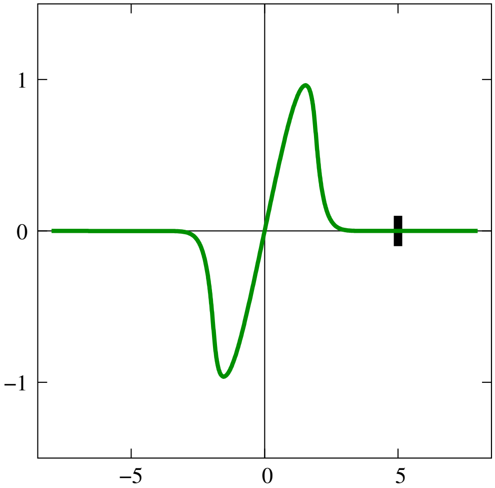
E.F
1
0
−1
−5
0
5

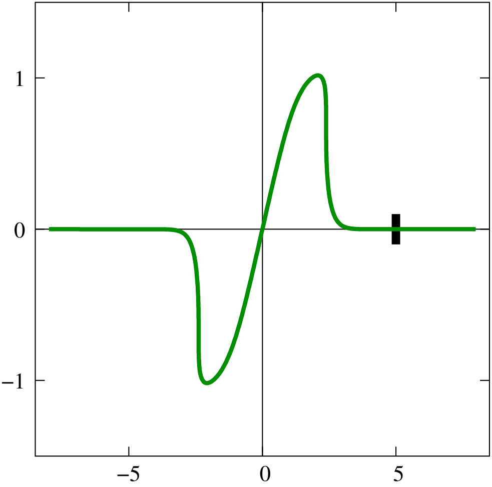

1
0
−1
−5
0
5

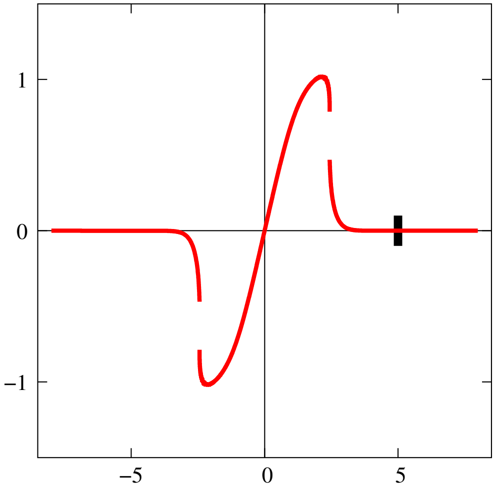

1
0
−1
−5
0
5

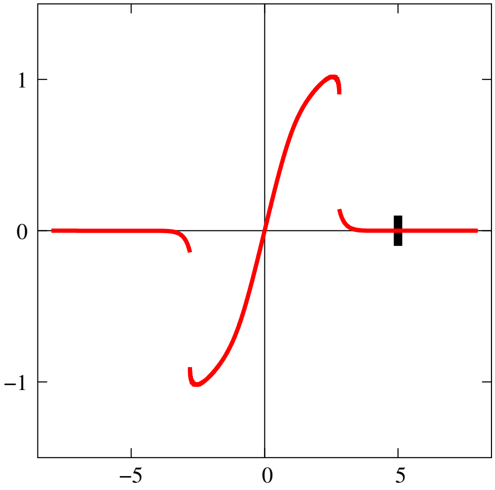

1
0
−1
−5
0
5

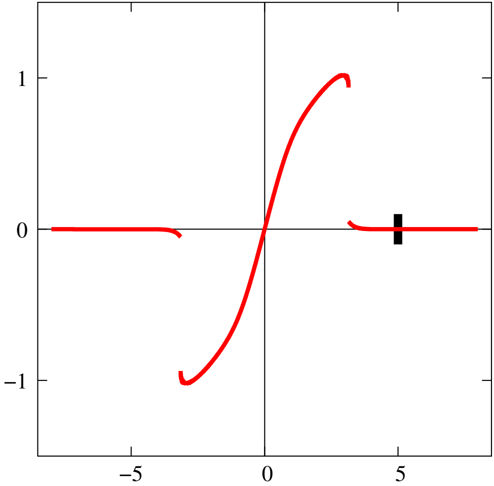

1
0
−1
−5
0
5

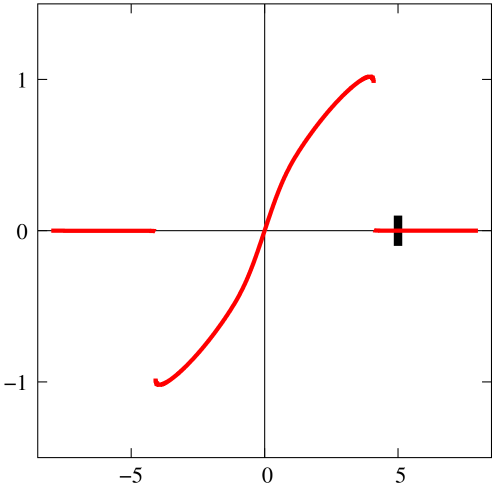

1
0
−1
−5
0
5

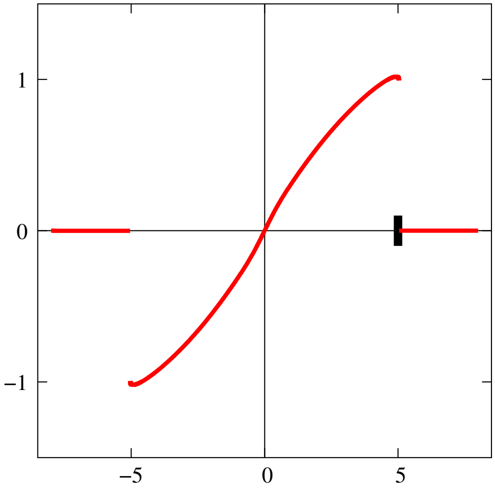

1
0
−1
−5
0
5

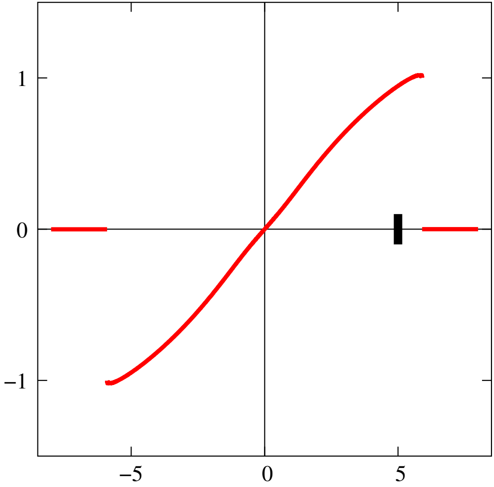

1
0
−1
−5
0
5


1
0
−1
−5
0
5


1
0
−1
−5
0
5


1
0
−1
−5
0
5


1
0
−1
−5
0
5


1
0
−1
−5
0
5

Inverse characteristics worked well for the particular system.

Warped time made things easier.

Shock discontinuities were easy to identify and calculate.

Newton-Raphson was the only numerical method used.

Soft analysis hides more than it reveals.

Generalization will not make a problem go away.

Compute before you theorize before you compute.

When you show the moon to a child,
it will only see your finger.

Thank you.

May the force be with you!

Dziwa, S. & Sauer, N. Waves without the wave equation: Examples from nonlinear acoustics. *Int. J. Eng. Sci.*, **133** (2018), 196–209.

Sauer, N. An excursion into classical nonlinear acoustics. *Int. J. Eng. Sci.*, **48**(2010), 670–684.

Sauer, N. The dynamic piston problem in classical nonlinear acoustics. *Math. Models and Meth, Appl. Sci.*, **21**(2011), 149–167.

# Index