## Impurity-induced stabilization of solitons in arrays of parametrically driven nonlinear oscillators

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Chains of parametrically driven, damped pendula are known to support soliton-like clusters of in-phase motion. For sufficiently large driving amplitudes the solitonic clusters become unstable and seed spatio-temporal chaotic states. We show that the pinning of the soliton on an attractive impurity (a pendulum which is longer than other pendula in the chain) expands dramatically its stability region and prevents the spatio-temporal chaos from emerging. We also show that impurities may serve as centres of spontaneous nucleation of solitons. Finally, stationary solitons pinned on impurities may develop spontaneous oscillations which subsequently synchronise. (The below figure shows synchronised oscillations of solitons pinned on two impurities placed symmetrically on a ring.)

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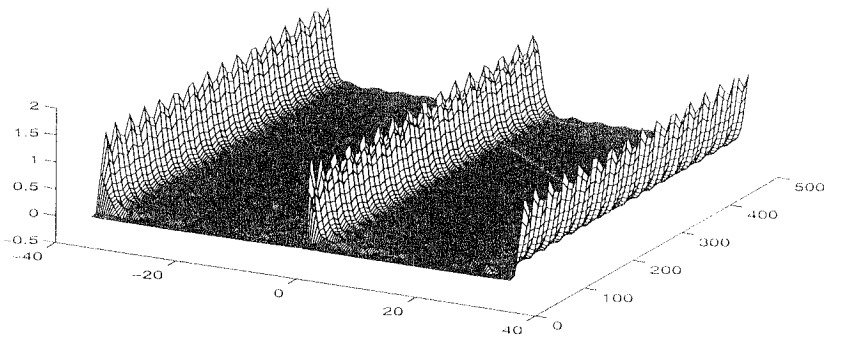




# SANUM99

at

Department of Applied Mathematics
University of Stellenbosch
South Africa

March 29 – 31, 1999

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PROGRAM

| | Monday | Tuesday | Wednesday |
|---|---|---|---|
| 08:00–08:45 | Registration | | |
| 08:50–09:00 | Opening | | |
| 09:00–09:50 | Nick Trefethen | Andre Weideman | Daya Reddy |
| 10:00–10:20 | Schalk Schoombie | Carl Rohwer | Eeva Rapoo |
| 10:30–11:00 | Tea | Tea | Tea |
| 11:00–11:50 | Anne Trefethen | Karin Goosen/BirgitRohwer | Nick Trefethen |
| 12:00–12:20 | Neil Muller | Willy Theron | Meeting |
| 12:30–14:00 | Lunch | Lunch | Lunch |
| 14:30–14:50 | Roumen Anguelov | | |
| 15:00–15:20 | Dana Murray | Alna van der Merwe | |
| 15:00–15:20 | Susan de Swardt | Lizette Zietsman | |
| 15:30–16:00 | Tea | Tea | |
| 16:00–16:20 | Valery Shchesnovich | Ruby van Rooyen | |
| 16:30–16:50 | Hanno Coetzer | Jacquie Diatezua | |
| 19:00–????? | | Conference Dinner | |



## Growth and decay in random recurrence relations

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What happens to the behavior of the Fibonacci recurrence if you randomize the signs at each step? The answer is that with probability 1, the sequence grows exponentially at the rate 1.13198824.... This is just the beginning of results that can be computed numerically for various problems with the flavor of random recurrences. The numerics is nontrivial, involving matrix problems in some cases of dimensions in the millions. The numbers in question are Lyapunov constants for certain Markov processes associated with random products of matrices, and there are connections with condensed matter physics, among other fields. The results described in this talk are due to Viswanath, Embree, Wright, and the author.

# APPROXIMATION OF BROWNIAN PATHS AND SDE SOLUTIONS: The good, the bad and the ugly

Dr E. Rapoo
UNISA[1]

We will to consider simple stochastic differential equations (SDEs) of the form

$$dY_t = \sum_{i=1}^{n} f_i(Y_t) dB(t)_i$$

where $B = (B_1, \ldots, B_n)$ is an $n$-dimensional Brownian motion. Numerical analysis of this kind of a system is surprisingly tricky if pathwise solutions are requested. The problems are caused by the special features of stochastic calculus, such as for instance the existence of infinitely many stochastic integrals, the unbounded variation of almost all Brownian paths, the "stability" problem between ODEs and SDEs and the martingale construction of stochastic integrals which means that path-dependent (eg. variable stepsize) methods must be used with extreme care. All these problems also appear in the related problem of trying to approximate the Brownian motion with a bounded-variation function and thus replace the SDE by an ODE.

We will utilize certain new insights into the nature of SDEs to express general guidelines about what is good and what is bad in numerical stochastic analysis and Brownian approximations, and will also suggest some new (ugly but useful?) tactics.



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# Eigenvalues, pseudospectra, and dynamics

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A new tool has become popular in the 1990s for the analysis of problems involving non-hermitian matrices and operators: pseudospectra, the regions in the complex plane bounded by levels curves of the norm of the resolvent. A survey is given of pseudospectra, their numerical computation, and their dynamical significance in fluid mechanics and other fields.

# Spurious behavior of difference schemes of various orders for the Korteweg-de Vries equation

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## Extended Abstract

In this talk we describe some interesting results which followed out of a discrete multiple scales analysis of the Korteweg-de Vries (KdV) equation. Details of the analysis itself were published before[1, 2, 3].

For the purpose of this talk we consider the KdV equation in the form

$$u_t + \eta u_x + \zeta u u_x + \gamma u_{xxx} = 0, \tag{1}$$

where the subscripts denote partial differentiation as usual and $\eta$,$\zeta$ and $\gamma$ are constants, with $\gamma \neq 0$. Furthermore we assume that suitable initial data

$$u(x,0) = \epsilon f(x), f(x) = O(1), \tag{2}$$

be prescribed, where $\epsilon$ is a small, real, positive number. We also enforce the following periodicity conditions

$$u(x \pm L, t) = u(x,t), f(x \pm L) = f(x), t > 0, x \in \Re. \tag{3}$$

We perform our discrete multiple scales analysis on the following semi discrete version of (1):

$$\partial_t u_j + \eta D^{1,p} u_j + (\zeta/3)(D^{1,p}(u_j)^2 + u_j D^{1,p} u_j) + \gamma D^{3,p+1} u_j, \tag{4}$$

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# Affine-Approximate Finite Element Methods

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We present an analysis to show that, for quadrilateral elements in two dimensions and hexahedral elements in three, it is possible to construct stable, convergent and accurate finite element approximations based on affine figures (parallelograms or parallelipipeds) that are 'close' to the original figures, in a sense made precise. The method has numerous computational advantages.

# Finite Element Method of the One–dimensional Problems arising in Non–Newtonian Flows

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Abstract

One–dimensional problems araising from unidirectional and radial flows, are studied. The hyperbolic nature of the problem necessitates a modification of the standard Galerkin–based finite element method; here the discontinuous Galerkin method is used for radial flows. The Galerkin method, the streamline upwind petrov–Galerkin (SUPG) method and the Streamline Upwind are analysed for the unidirectional flow, relative accuracy of these methods is discussed.



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where $D^{m,p}$ indicates a central difference approximation for the m-th derivative with respect to $x$, with stencil width of $2p+1$ grid points. These central differences were calculated using an algorithm of Fornberg[4].

When performing a a multiple scales analysis on (1), removal of secular terms lead to a form of the cubic Schrödinger equation. Effectively this equation describes the nonlinear behaviour of the amplitude of a slowly modulated monochromatic wave, i.e. one approximately of the form

$$u(x,t) \approx V(X_1, T_1, T_2)e^{i\theta} + \text{ complex conjugate} \tag{5}$$

where

$$\theta = khj - \Omega T_0, \tag{6}$$

with $k$ the wave number of the carrier wave, $\Omega$ the carrier frequency, and where

$$X_1 = \epsilon hj, T_0 = t, T_1 = \epsilon t, T_2 = \epsilon t^2$$

are the various scales in space and time.

When performing a discrete version of this analysis (described in [2, 3]), a discrete version of the cubic Schrödinger equation is obtained, but only for those carrier wave numbers for which a certain function $g(h, k, \eta, \gamma))$ of grid length, wave number, and the parameters $\eta$ and $\gamma$ is not zero.

We investigated cases where $g = 0$, and found by means of numerical experiments that it leads to spurious behaviour, in the sense that a significant spurious wave mode is created, and that there are spurious modifications to the amplitudes of the real solution.

What was interesting was that, as higher order difference methods were used, these spurious effects tended to become smaller, especially in the case of lower wave numbers.

It seems that for sufficiently high order methods, this particular type of spurious behaviour would become insignificant.

## References

[1] S.W. Schoombie, *J. Comp. Phys.* **101**, 55 (1992)

[2] S.W. Schoombie and E. Marè, "A Generalized Discrete Multiple Scales Analysis Technique" in *Numerical Analysis*, edited by D.F. Griffiths and G.A. Watson, World Scientific, Singapore (1996)

[3] E. Marè and S.W. Schoombie, "A Review of Multiple Scales Techniques Used for Perturbation Analysis of Numerical Approximation Methods" in *Differential Equations and Chaos*, edited by N.H. Ibragimov, F.M. Mahomed, D.P. Mason and D. Sherwell, New Age International Publishers (1996)

[4] B. Fornberg, *Math. Comp.* **31**, 699 (1988)

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## The automated detection of gravitational lensing effects.

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In the search for extra solar planets using the Gravitational Microlensing technique, a more efficient way of data analysis is needed. A number of different techniques are available - all of them far from perfect. One of the main goals is to improve on these existing techniques considering the difficulties that arise with the dimness of lensing objects, atmospheric conditions, optical and instrumental aberrations, upper and lower detection limits and interference such as cosmic rays. Most of these difficulties are variable and can be predicted with some difficulty and only by approximation. Different conditions of observation also give different orientations and different stellar shapes due to seeing changes. In this talk we describe the basic problems and discuss triple correlation in some detail.

# 6 Results: The eigenvalue problem

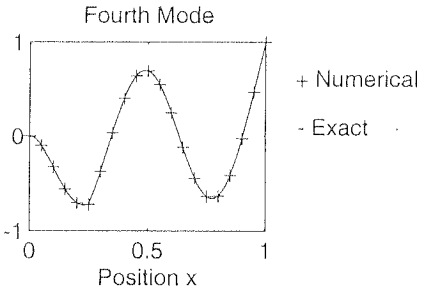
Natural frequencies.

| | $\delta = 0.1$ and $\alpha = 0.5$ | | $\delta = 0.1$ and $\alpha = 0.25$ | |
|---|---|---|---|---|
| | Characteristic eq | FEM ($n = 20$) | Characteristic eq | FEM ($n = 20$) |
| 1 | 1.85398229 | 1.85398233 | 1.80171085 | 1.80171089 |
| 2 | 4.48885757 | 4.4888615 | 4.68669099 | 4.68669587 |
| 3 | 7.85450288 | 7.85456715 | 7.63492665 | 7.63498240 |
| 4 | 10.58303254 | 10.58331557 | 10.69658787 | 10.69688642 |
| 5 | 14.13716764 | 14.13836503 | 14.07427374 | 14.07544299 |
| 6 | 16.70575901 | 16.70847862 | 17.17565008 | 17.17879279 |
| 7 | 20.42035227 | 20.42773454 | 19.81144131 | 19.81774005 |
| 8 | 22.87313678 | 22.87696401 | 23.03705626 | 23.05027743 |
| 15 | 45.55309347 | 45.90162370 | 44.96825963 | 44.90613713 |

(Results given to eight decimal places.)

In the table above we compare approximations for the natural frequencies obtained by a so called "exact method" to the FEM. Using 20 elements the results for the first 15 frequencies are excellent.

The FEM enables us to approximate the modes. For example, let $\delta = 0.1$ and $\alpha = 0.25$:




As explained in Part 1, it is only feasible to calculate a few modes using the "exact method". This constraint is not experienced with the FEM.

# 7 References

[S] G Strang & GJ Fix, *An analysis of the finite element method*, Prentice-hall, New Jersey, 1973



Libraries of numerical algorithms.

Anne Trefethen

NAG has been developing and supporting numerical software libraries for nearly thirty years. In that time there been huge advances not only in numerical algorithms, but also in hardware and software environments. In this talk I will describe some of the changing issues involved in numerical library development and some future directions as we see them.

# A Fast Method for Updating Eigenfaces

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Abstract

It is well known that, given a set of images $\mathrm{I}_n$, $n \in [1, M]$ that using Principal Component Analysis (PCA), we can find a basis with dimension $< M$ for this set which is optimal in the sense that the error made by representing a image in the training set using this basis is as small as possible. These principal components are known as the eigenfaces and are the eigenvectors of the covariance matrix $C = \frac{1}{M}\sum_{i=1}^{M} \frac{(\mathrm{I}_n - \mathrm{A})(\mathrm{I}_n - \mathrm{A})^T}{\|\mathrm{I}_n - \mathrm{A}\|^2}$ where $A = \frac{1}{M}\sum_{n=1}^{M} \mathrm{I}_n$.

It has been shown that these eigenfaces provide a good method to represent faces for purposes of recognition, provided that the initial training set is sufficiently representative. However, in many situations, this is not true. If presented with a new face which cannot be adequately represented by the eigenfaces, we need to calculate a new set of eigenfaces which includes information about the new face.

While it is possible to create a new covariance matrix and recalculate the eigenvectors, this approach is inefficient if we are dealing with large training sets. We present a fast method for calculating a good approximation to the eigenvectors of the new covariance matrix using only the new face and the already calculated eigenfaces. We can rewrite the problem in terms of finding the eigenvectors of an extremely sparse matrix with a well determined structure. This allows us to calculate the new eigenvectors extremely efficiently.



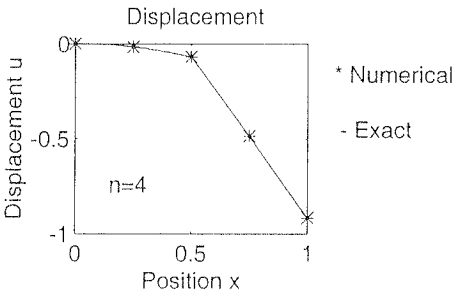
Since the basis functions take the derivative into account, excellent results are obtained using this interpolate to approximate a function.

Thus we solve the system of linear equations

$$K\bar{u} = M\bar{f}.$$

## 5 Results: The equilibrium problem

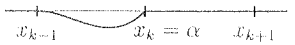
We use $f(x) = 5\sin(5x)$ for illustration purposes. For example, let $\delta = 3$ and $\alpha = 0.5$:



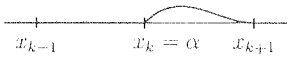

For as little as two intervals we already obtained excellent approximations.

| Number of subintervals | $\max\|u - u^h\|$ | $\max\|u' - (u^h)'\|$ |
|---|---|---|
| 2 | 0.0387 | 0.1003 |
| 4 | 0.0030 | 0.0384 |
| 10 | $7.9178 \times 10^{-5}$ | 0.0397 |
| 100 | $5.7216 \times 10^{-9}$ | 0.04 |

The value of $\delta$ used above is excessive and even better results are obtained for smaller values of $\delta$.




and




This choice of basis functions makes provision for the handling of functions with discontinuous derivatives at $x_k = \alpha$.

# 4 The bending matrix

The bending matrix, $K$, for an undamaged beam is defined by $K_{ij} = b(\phi_j, \phi_i)$ for $1 \leq i, j \leq 2n + 2$.

In modifying the bending matrix for an undamaged beam to the bending matrix for a damaged beam, two aspects must be taken into account:

- one of the Type 2 basis has changed and
- the bilinear form has an additional term.

By replacing the row and column associated with the Type 2 basis function at $x_k$ in the bending matrix of the undamaged beam by two rows and columns respectively, provision is made for the additional basis function. The values in the matrix in these two rows and columns have to be modified.

Only four elements in the bending matrix will change due to the additional term, $\left(u^{h'}(\alpha^+) - u^{h'}(\alpha^-)\right)\left(v'(\alpha^+) - v'(\alpha^-)\right)/\delta$, in the bilinear form.

The boundary conditions $v(0) = v'(0) = 0$ are accommodated by removing appropriate rows and columns.

The components of $\bar{u}$ yield the function values of $u^h$ at nodes 2 to $n + 1$, the value of the derivative of $u^h$ at nodes 2 to $k - 1$ and $k + 1$ to $n + 1$, and the left and right derivatives of $u^h$ at node $k$.

The vector $\bar{F}$ in the matrix form of the Galerkin approximation, can be approximated using numerical integration. Since the mass matrix $M$, with $M_{ij} = (\phi_j, \phi_i)$, is required for the eigenvalue problem, we rather use the interpolate of $f$ to calculate $\bar{F}$.

We define the interpolate $f_I$ as

$$f_I = \sum_{i=1}^{n+1} f(x_i)\phi_i + \sum_{i=n+2}^{n+k} f'(x_i)\phi_i + f'(\alpha^-)\phi_{n+k+1} + f'(\alpha^+)\phi_{n+k+2} + \sum_{i=n+k+3}^{2n+3} f'(x_i)\phi_i.$$



## On the Mathematical Foundation of the Nonstandard Finite Difference Method

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The finite difference method is one of the oldest, simplest and thus very popular technique for the numerical treatment of differential equations. For most of the equations in mathematical physics finite difference schemes have been designed and investigated from both theoretical point of view (convergence) and practical point of view (consistency, stability). Furthermore, provided the solution is smooth enough, these schemes produce numerical solution with very good asymptotic behaviour when the step-size approaches zero.

One disadvantage of this standard approach is that qualitative properties of the exact solution are not transferred to the numerical solution. In practice, the limit is not reached. What we have is the numerical solution obtained for one or several values of the step-size. Thus, the stated disadvantage might be catastrophic.

The nonstandard approach discussed in this talk preserves essential properties of the exact solution. This is achieved by replacing derivatives by nonstandard finite difference operators with denominators that are suitable function of the step-size, nonlinear expressions being approximated nonlocally.

Nonstandard finite difference techniques were developed empirically for solving practical problems in applied sciences and engineering. Although they produce results satisfactory to their users, these techniques have not yet been subjected to rigorous mathematical analysis. Some concepts are still unclear.

Our aim is to give rigorous mathematical meaning and justification of some key concepts involved in the design of nonstandard finite difference schemes. We also discuss a certain number of "successful" empiric procedures the mathematical justification of which is still pending.

# Numerical Mathematics Issues in Financial Engineering

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Financial derivatives, or options, are introduced and the numerical methods used for options pricing are discussed. When these methods, typically based on a discretization of the state space, are generalized to higher dimensions, the curse of dimensionality implies that the computational burden grows exponentially with the state dimension.

Differential dynamic programming (DDP) is discussed as an example of a dynamic programming method that is based on a parameterization of the state space that overcomes the curse of dimensionality. While DDP is not directly applicable to the stochastic problems of options pricing, DDP may point the way to parameterizations that are more suitable for higher dimensional finance problems.



$B(u^h, v) = b(u^h, v) + \frac{1}{\delta}\left((u^h)'(\alpha^+) - (u^h)'(\alpha^-)\right)(v'(\alpha^+) - v'(\alpha^-)) = (f, v)$ *for all* $v \in S$.

The Galerkin approximation reduces to a system of linear equations of the form

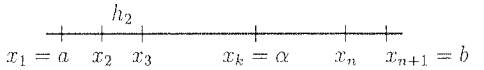
$$K\vec{u} = \vec{F}$$

with $K_{ij} = B(\phi_i, \phi_j)$, $F_i = (f, \phi_i)$ and $u^h = \sum_{i=1}^{m} u_i \phi_i$.

## 3 Basis functions

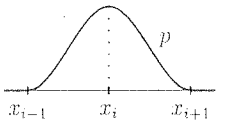
Hermite piecewise cubics ([S], p. 56) are successfully used for the Galerkin approximation associated with an undamaged beam. The possibility of adapting the same basis functions for the case where we have damage is investigated.

The interval $[a, b]$ is divided into subintervals of length $h_i$ for $i = 1, 2, \ldots n$ in such a way that $\alpha$ corresponds to a node say $x_k$.

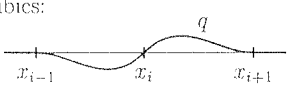



With each node we associate two Hermite piecewise cubics which we will refer to as Type 1 and Type 2 basis functions.

Type 1 piecewise cubics:




Type 2 piecewise cubics:




For the problem of a beam with damage we associate two basis functions of Type 2 with $x_k = \alpha$ instead of the single classical Type 2 basis function:

# A numerical study of the vibrations of a damaged beam. Part II: Application of the FEM.

L Zietsman (UNISA), NFJ van Rensburg (UP), AJ van der Merwe (UP)

## 1 Introduction

In Part 1 a mathematical model for the vibrations of a cantilever beam damaged at a single point is given. We consider the implementation of the finite element method for the equilibrium problem as well as the eigenvalue problem.

The variational formulation for the equilibrium problem is given in Part 1 and the Galerkin approximation to this is formulated in Section 2 of Part 2.

In Section 3 Hermite piecewise cubic basis functions are modified to deal with the discontinuity in the derivative. The construction of the bending matrix is discussed in Section 4.

Numerical results obtained using the finite element method for the equilibrium problem and the eigenvalue problem are given and compared to exact solutions in Sections 5 and 6 respectively.

## 2 The Galerkin approximation

In order to define the Galerkin approximation for the variational form, choose a finite number of functions from the set of test functions, $T(I) = \{v \in C[0,1] \,\big|\, v|_{[0,\alpha]} \in C^2_+[0,\alpha],\ v|_{[\alpha,1]} \in C^2_+[\alpha,1],\ v(0) = v'(0) = 0\}$, say $\phi_1,\ \phi_2,\ \ldots,\ \phi_m$ and set $S = \text{span}\{\phi_1,\ \phi_2,\ \ldots,\ \phi_m\}$.

The Galerkin approximation for the problem in variational form is formulated as:

*find* $u^h \in S$ *with*





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Trapezoidal rules with endpoint corrections can often be considered as an alternative to Newton-Cotes rules in numerical integration. A specific example is the Gregory rule of order two, i.e. the Lacroix rule, which has as its analogue in Newton-Cotes quadrature the well-known Simpson rule, both rules being exact on cubic polynomials. One of the criterions which can be used in the comparison of different quadrature rules is to consider the relative sizes of the error constants corresponding to integrands $f$ in the relevant continuity classes. We employ here a quadratic nodal spline interpolation method to compute sharp error constants for the Lacroix rule, which are then shown to compare favourably with their Simpson rule counterparts. In addition, a generalized Gregory rule based on non-equally spaced abscissas is explicitly constructed.

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# Soliton-radiation resonance in the parametrically driven, damped NLS equation

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The parametrically driven, damped NLS equation reads

$$iq_t + q_{xx} + 2|q|^2 q = -i\gamma q + he^{i\Omega t}\bar{q}, \qquad (1)$$

where $q$ is the complex amplitude (bar denotes complex conjugation), $\gamma$ is the damping coefficient, $h$ is the driving strength, and $\Omega$ is the driving frequency. Eq. (1) describes, for instance, the nonlinear Faraday resonance in a vertically oscillating water trough, the parametric generation of the spin waves in the one-dimensional ferro and anti-ferromagnets, and the effect of the phase sensitive parametric amplifiers on the solitons propagating in optical fibers. It also appears in the theory of waves in plasma. The NLS equation, i. e., (1) with the r.h.s. equal to zero, is integrable by the Inverse Scattering Transform method (IST), its solution consists of two mutually noninteracting parts: solitons and radiation. The driven, damped NLS equation also has one-soliton solution. Consider a more general solution to (1) which is comprised of the soliton and radiation via the perturbation theory based on the IST. Being stable "equilibrium" modes of the integrable NLS equation, in the first order of the perturbation theory both the soliton and radiation give rise to corresponding oscillating solutions to (1). The soliton brings one oscillation mode with the frequency growing from zero as $h$ increases, while the radiation part of the solution brings infinitely many oscillation modes with



We define a bilinear form $B(\ldots)$ by

$$B(u,v) := b(u,v) + \frac{1}{\delta}(u_2'(\alpha) - u_1'(\alpha))(v_2'(\alpha) - v_1'(\alpha)) \text{ for all } u, v \in C_+^2.$$

**Variational formulation for the equilibrium problem**

*For $f \in L^2$ find $u \in T(I)$ such that $B(u,v) = (f,v)$ for all $v \in T(I)$.* (2)

The following result deals with the equivalence of the boundary value problem and the variational formulation.

**Theorem 1** *If $u = \langle u_1, u_2 \rangle$ is a solution of* (2) *and $u \in T(I) \cap C^4$, then $u$ is a solution of the boundary value problem.*

**Theorem 2** *The variational problem* (2) *has at most one solution.*

For computational purposes we formulate the variational problems differently. The restriction of a function $v$ to an interval $[a,b]$ is denoted by $v|_{[a,b]}$. Define the set of test functions by

$$T(I) = \{v \in C[0,1] \,\big|\, v|_{[0,\alpha]} \in C_+^2[0,\alpha] \text{ and } v|_{[\alpha,1]} \in C_+^2[\alpha,1]\}.$$

**Variational formulation for the equilibrium problem**

*For $f \in L^2$ find $u \in T(I)$ such that $B(u,v) = (f,v)$ for all $v \in T(I)$.* (3)

**Variational formulation for the eigenvalue problem**

*Find $u \in T(I)$ and a complex number $\lambda$ such that*

$$B(u,v) = \lambda(u,v) \text{ for all } v \in T(I). \qquad (4)$$

## References

[VvR ] Viljoen, HJ & van Rensburg, NFJ, Damage Detection in Composites by ZnO Sensors,AIChE Journal, **42**, no 4, April 1996, 1101-1107.

Let $I = (0,1)$, $I_1 = (0,\alpha)$ and $I_2 = (\alpha,1)$. Define the following product spaces:

$$\begin{aligned} L^2 &:= L^2(I_1) \times L^2(I_2), \\ C^i &:= C^i(\bar{I}_1) \times C^i(\bar{I}_2),\ i = 0,1,\ldots \\ C^i_+ &:= C^i_+(\bar{I}_1) \times C^i_+(\bar{I}_2),\ i = 0,1,\ldots \\ C^\infty_0 &:= C^\infty_0(I_1) \times C^\infty_0(I_2). \end{aligned}$$

As usual $L^2(I_j)$ refers to the space of square integrable functions, $C^i(\bar{I}_j)$ to the space of functions with continuous derivatives on $\bar{I}_j$ up to order $i$ and $C^\infty_0(I_j)$ to the space of infinitely differentiable functions with compact support in $I_j$. $C^i_+(\bar{I}_j)$ denotes the space of functions with continuous derivatives on $\bar{I}_j$ up to order $i-1$ and with a piecewise continuous derivative of order $i$. At the points of discontinuity of the $i$-th derivative, both the left and right derivative should exist and be finite. The space $C^2_+$ is contained in the Sobolev space $H^2(I)$.

$L^2$ is a Hilbert space with the inner product $(.,.)$ given by

$$(u,v) = \int_0^\alpha u_1 v_1 + \int_\alpha^1 u_2 v_2 \text{ for all } u, v \in L^2.$$

We define the differential operator $L$ by

$$Lu := \langle u_1^{(4)}, u_2^{(4)} \rangle \text{ for all } u \in C^4.$$

We define a bilinear form $b(.,.)$ by

$$b(u,v) := \int_0^\alpha u_1'' v_1'' + \int_\alpha^1 u_2'' v_2'' \text{ for all } u, v \in C^2_+.$$

Integration by parts yields the following Green formula.

**Lemma 1** *For any $u \in C^4$ and $v \in C^2_+$*

$$(Lu, v) = b(u,v) + [u_1''' v_1 - u_1'' v_1']_0^\alpha + [u_2''' v_2 - u_2'' v_2']_\alpha^1.$$

We define a set of test functions $T(I)$ as

$$T(I) := \{v \in C^2_+ : v_1(0) = v_1'(0) = 0, v_1(\alpha) = v_2(\alpha)\}.$$

Note that $u = \langle u_1, u_2 \rangle \in T(I)$ is well-defined as a function on $\bar{I}$ as $u_1(\alpha) = u_2(\alpha)$. Also, that these test functions have continuous derivatives on $\bar{I}_1$ and $\bar{I}_2$ but that a discontinuity in the derivative may occur at $x = \alpha$.



the frequencies bounded from below. With the help of the IST technique one can formulate a standard scalar eigenvalue problem for the frequencies of the oscillation modes of radiation. Besides a continuous spectrum of radiation modes, which is nothing but the deformed spectrum of the integrable NLS equation, there are discrete modes also, whose frequencies detach from the lower boundary of the continuous spectrum and go down with growth of $h$, while the boundary of the continuous spectrum itself goes up. Due to soliton-radiation interaction, there is some critical value of $h$, when the eigenfrequencies of the system describing the soliton coupled to the lowest discrete radiation mode come in resonance. This soliton- radiation resonance results in the soliton instability.

# ON THE SEGMENTATION OF STATIC HANDWRITTEN SIGNATURES AND THE ENHANCEMENT OF THE PERFORMANCE OF AN ON-LINE SIGNATURE VERIFICATION SYSTEM

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**Abstract:** We continue to investigate the feasibility of using the Radon Transform and a dynamic programming algorithm to authenticate handwritten signatures on cheques. Our current system achieves an equal error rate of approximately 23% when only very high quality forgeries (skilled forgeries) are considered and an equal error rate of approximately 10% in the case of only casual forgeries.

We now address two of the major deficiencies of our current system. All the projections in the Radon Transform are considered to be of equal importance and only the Karhunen-Loève Transform is used to ensure rotation invariance. Both of these deficiencies are addressed by first finding suitable mappings between the projections of a test and training signature, before comparing them. Projections that have similar statistical properties are



Let $\lambda = \mu^4$. For an undamaged beam the natural frequencies are given by $\mu_i^2$ where $\mu_i$ is calculated from the equation

$$\cos\mu\cosh\mu = -1.$$

The associated mode of vibration is of the form

$$w_i(x) = A\sin(\mu_i x) - A\sinh(\mu_i x) + B\cos(\mu_i x) - B\cosh(\mu_i x)$$

where the ratio between $A$ and $B$ is determined by the boundary conditions at $x = 1$. For the damaged beam the form of the mode is given by

$$w(x) = \begin{cases} A\sin(\mu x) - A\sinh(\mu x) + B\cos(\mu x) - B\cosh(\mu x) & \text{for } 0 < x < \alpha \\ (C+A)\sin(\mu x) + (D-A)\sinh(\mu x) & \\ \qquad +(E+B)\cos(\mu x) + (F-B)\cosh(\mu x) & \text{for } \alpha < x < 1, \end{cases}$$

where the boundary conditions at $x = 0$ have already been taken into account.

From the continuity conditions and the jump condition at $x = \alpha$, the constants $C$, $D$, $E$, and $F$ can be expressed in terms of $A$ and $B$. Finally, from the two boundary conditions at $x = 1$, the characteristic equation for the natural frequencies can be constructed. Solving this equation numerically yields the natural frequencies for the damaged beam and for each natural frequency a vibration mode can then be obtained. As is expected, this approach yielded only the first few natural frequencies and vibration modes as the cosh and sinh functions are difficult to handle numerically.

In order to find higher natural frequencies and associated modes a finite element method was used. The natural frequencies and modes calculated from the characteristic equation, were used to establish the accuracy of the finite element calculations.

# 4 Variational formulation of problems

In order to find variational formulations for the problems in Section 2, the functions $u(.)$ and $u(.,t)$ will be presented as pairs $u(.) = \langle u_1(.), u_2(.)\rangle$ and $u(.,t) = \langle u_1(.,t), u_2(.,t)\rangle$ with $u_1$ the restriction of $u$ to the interval $[0,\alpha]$ and $u_2$ the restriction to $[\alpha,1]$. In terms of this notation $u'(\alpha^-) = u_1'(\alpha)$ and $u'(\alpha^+) = u_2'(\alpha)$. For the equilibrium problem the conditions at $x = \alpha$ become:

$$\begin{aligned} u_1(\alpha) &= u_2(\alpha), \\ u_1'''(\alpha) &= u_2'''(\alpha), \\ u_1''(\alpha) &= u_2''(\alpha) = \frac{1}{\delta}(u_2'(\alpha) - u_1'(\alpha)). \end{aligned}$$

interval $I = [0, 1]$ and the weak point is located at a fixed point $x = \alpha$. The displacement of point $x$ at time $t$ is $u(x, t)$.

A mathematical model for this problem is given by:

$$\begin{aligned}
\partial_t^2 u &= -\partial_x^4 u, \quad 0 < x < 1, \ x \neq \alpha, \ t > 0, \qquad (1)\\
u(0,t) &= \partial_x u(0,t) = 0,\\
\partial_x^2 u(1,t) &= \partial_x^3 u(1,t) = 0,\\
u(\alpha^+,t) &= u(\alpha^-,t),\\
\partial_x^i u(\alpha^+,t) &= \partial_x^i u(\alpha^-,t), \ i = 2,3,\\
\partial_x^2 u(\alpha,t) &= \frac{1}{\delta}(\partial_x u(\alpha^+,t) - \partial_x u(\alpha^-,t)),\\
u(x,0) &= a(x),\\
\partial_t u(x,0) &= b(x).
\end{aligned}$$

The problem is formulated in nondimensional form with $\delta$ a positive constant. Right and left limits and right and left derivatives are denoted by the superscripts $+$ and $-$.

The conditions at $x = \alpha$ model the damage to the beam. Note that the magnitude of $\delta$ indicates the extent of damage at a fixed point $x = \alpha$ and that $\delta = 0$ corresponds to a uniform beam with no damage.

We also formulate an equilibrium problem for the beam. In this case a distributed load $f$ is introduced. and the transverse displacement of point $x$ in $[0, 1]$ is $u(x)$.

# 3 Natural frequencies and modes of vibration

One way to calculate the natural frequencies and modes of vibration of the damaged beam is to apply the method of separation of variables to (1) and to solve the following resulting eigenvalue problem:

$$\begin{aligned}
w^{(4)} - \lambda w &= 0, \quad 0 < x < 1, \ x \neq \alpha,\\
w(0) &= w'(0) = w''(1) = w'''(1) = 0,\\
w(\alpha^+) &= w(\alpha^-),\\
w''(\alpha^+) &= w''(\alpha^-),\\
w'''(\alpha^+) &= w'''(\alpha^-),\\
w''(\alpha) &= \frac{1}{\delta}(w'(\alpha^+) - w'(\alpha^-)).
\end{aligned}$$



mapped and assigned a suitability value based on these properties. Mapped projections are compared and those with high suitability vales are preferred.

These ideas also form the framework for a segmentation algorithm that successfully separates cursive handwritten signatures at their perceptually important points.

## Numerical Inversion of the Laplace Transform

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The problem of inverting the Laplace Transform numerically has generated enormous interest over the last two or three decades, and dozens of methods have been proposed. These methods can be divided into two classes: The first class of methods is aimed at the situation where only real values of the transform are available. The second class of methods, which this talk will focus on, consists of those methods that assume the transform to be known as an analytic function that can be evaluated at arbitrary points in the complex plane. In this case the problem reduces to the numerical computation of the complex inversion formula known as the Bromwich integral. In the latter class there are three methods that have surfaced to the top in numerous empirical studies: the direct evaluation of the Bromwich integral plus sequence acceleration, Talbot's method, and Weeks's method. The first two methods are both based on the trapezoidal rule, the last on Laguerre expansions. In this talk we shall survey these three methods, and discuss recent advances in the selection of the free parameters that control the accuracy of these methods.



# A numerical study of the vibrations of a damaged beam. Part I: Theory

L Zietsman, NFJ van Rensburg, AJ van der Merwe

## 1 Introduction

The detection of damaged regions in composite materials is attracting much interest. See [VvR] for details and numerous other references. Different models are used to describe damage in a beam or plate. We will consider the model proposed in [VvR]. The damage to the beam is modelled as an elastic joint. The model is given in Section 2. The natural frequencies for the damaged beam are calculated from the characteristic equation obtained from the associated eigenvalue problem. Details are given in Section 3. As is well-known this approach yields only the first few natural frequencies and modes.

In this study we develop a finite element method for calculating the natural frequencies and modes. The main problems are to find a class of test functions which can deal with the discontinuity in the derivative that appears as a result of the elastic joint and subsequently to construct suitable basis functions for the finite element computation.

In Section 4 we indicate that our choice of test funcions yields a well-defined variational problem.

The construction of suitable basis functions for the finite element method and a discussion of numerical results will follow in the second part of this presentation.

## 2 Mathematical Model

We consider small one dimensional transverse vibrations of a uniform cantilever beam damaged at a single point. The reference configuration is the

# The hopping hoop

W F D Theron *

March 24, 1999

**Abstract**

We analyse the rolling motion of a light, rigid hoop with a massive particle fixed to the rim. The resulting motion is surprisingly complex; inter alia, we show that this is a "faster than gravity" model, and that the hoop will hop under the correct conditions. We also show that in the case of a massless hoop, the occurence of zero normal reaction does *not* imply hopping, thereby contradicting previously published opinions in this respect.

## References

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# The Babylonian Method and Higher Order Approximations to Square Roots.

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Interest in methods of calculating the sqare root of a number has resurfaced at various times and places. Bertrand Russel refered to an interesting method of the old Greeks, and Lothar Collatz referred to a Babylonian method, which is effectively a Newton iteration. Using these scant references and only basic mathematics, interesting derivations for these, and other methods. can be derived. The proccess is instructive and interesting. Some optimal properties are argued directly.

# Proposal: Wavelets analysis of Missile data

Birgit Rohwer and Karin Goosen*

March 11, 1999

## 1 Introduction

Our aim in this report is to introduce wavelet analysis tools to analise, evaluate and characterize missiles using data obtained from radar and other sensors.

To illustrate the various advantages of wavelets, we experimented with simulated data. We have made certain assumptions to simulate the path of a typical medium sized missile, launched from the ground. The path is graphically described by a graph of height versus time. We assume that the missile path is made up of a basic vertical-horisontal trajectory, with visible deviations. Piecewise continuous height accelerations as a result of linear "bang-bang" controls means that our basic path will be described mathematically by a quadratic spline. We account for the deviations, as follows:

- By adding vibrations of fixed frequency at random intervals,we describe the characteristic tracking instrument vibrations and aerodynamic and missile control natural vibrations.
- To account for statistical measuring errors we add Guassian noise.
- Due to glint caused by radar reflections, we add impulsive noise.

*Supervisor: CH Rohwer



With our wavelet analysis tools we illustrate the easy identification and extraction of the frequencies in the data, removal of noise and economical storage of data.

In this report we will suppose that the data from the simulated path, y, is of length 2n+1 and then continue to describe how a wavelet decomposition is executed on the data. The general step of the decomposition is described here, complete decomposition is obtained through repeated application of this step.