Earthquake induced oscillations of high rise buildings and other vertical structures

S Du Toit

Department of Mathematics and Applied Mathematics
University of Pretoria

March 2016

Supervisor: Prof NFJ van Rensburg
Co-supervisor: Dr M Labuschagne

NEES (Network for Earthquake Engineering Simulation), Simpson Strong-Tie and Colorado State University
Recent earthquakes have shown that damage in non-structural components and in building contents can have large economic consequences as well as safety and egress concerns. ... (2) typically more than 75% of the construction cost is associated with non-structural components; and (3) localized damage in certain non-structural systems can affect the functionality of large portions of the building.” - Reinoso and Miranda, 2005.
Need models to simulate effect of oscillations.

Tall buildings are often modelled as vertical beams.

[RM05] - 14 articles use beam models for buildings.

[RM05] - Building Seismic Safety commission and American Society of Civil Engineers use analytical studies and recovered data for safety specifications of new buildings.
Timoshenko model

- Rigorous derivation from three-dimensional linear elasticity presented in Cowper, 1966. Inspires confidence in the model.
- Stephen and Puchegger, 2006; Labuschagne, Van Rensburg and Van der Merwe, 2009 - Timoshenko theory compared to multi-dimensional model. Timoshenko theory is an excellent approximation in the case of beam applications, i.e. for transverse loads.
- Van Rensburg and Van der Merwe, 2006; [LVV09] - Timoshenko model compared to Rayleigh and Euler-Bernoulli models. These models can be useful when $\beta$ is large.
- Rayleigh and Euler-Bernoulli models are special cases of Timoshenko model.
Timoshenko model

Equations of motion:

\[ \rho A \partial_t^2 w = \partial_x V + Q, \]  
\[ \rho l \partial_t^2 \phi = V + \partial_x M, \] (1) (2)

The constitutive equations for the moment \( M \) and the shear force \( V \) are

\[ M = EI \partial_x \phi, \]  
\[ V = AG\kappa^2 (\partial_x w - \phi). \] (3) (4)
Dimensionless form of the **Timoshenko model**

\[
\begin{align*}
\partial_t^2 w &= \partial_x V + Q, \\
\frac{1}{\alpha} \partial_t^2 \phi &= V + \partial_x M, \\
M &= \frac{1}{\beta} \partial_x \phi, \\
V &= \partial_x w - \phi.
\end{align*}
\]

The boundary conditions for a cantilever beam are

\[w(0, t) = \phi(0, t) = 0\]

at the clamped end and

\[M(1, t) = 0 \text{ and } V(1, t) = 0\]

at the free end.
Rayleigh model

Assume that the cross section remains perpendicular to the neutral plane. This implies that $\partial_x w = \phi$.

\[
\partial_t^2 w = \frac{1}{\alpha} \partial_t \partial_x^2 w - \partial_x^2 M + Q, \tag{9}
\]
\[
M = \frac{1}{\beta} \partial_x^2 w. \tag{10}
\]

The boundary conditions are the same as for the Timoshenko beam except that $\partial_x w(0, t) = 0$ replaces $\phi(0, t) = 0$. 
Shear-T model

Han, Benaroya and Wei, 1999 consider four beam theories where in one shear is taken into account but not rotary inertia.

\[ \partial^2_t w = \partial_x V, \quad (11) \]
\[ 0 = V + \partial_x M. \quad (12) \]

The constitutive equations and boundary conditions are the same as for the Timoshenko model.
Stiffness parameter $\frac{1}{\beta}$

$$\beta = \frac{AG\kappa^2\ell^2}{EI} \left( \alpha = \frac{Al^2}{I} \quad \text{and} \quad \gamma = \frac{\beta}{\alpha} \right)$$

[VV06]; [LVV09] - Timoshenko model compared to Rayleigh and Euler-Bernoulli models. These models can be useful when $\beta$ is large.

- Depending on initial data / manner of excitation, value of $\beta$ between 300 and 1200 may be sufficient.
- For $\beta \approx 300$ fundamental frequency for these models is acceptable but not the higher frequencies.
- For $\beta < 100$ they should not be considered.
Modes of vibration

Natural frequencies of vibration is used to compare beam models. This approach was also used in

- [SP06] and [LVV09] - Timoshenko v.s. multi-dimensional model;
- [VV06] and [LVV09] - Timoshenko v.s. Rayleigh and Euler-Bernoulli.

For the modal analysis we follow [VV06].
Eigenvalue problem Timoshenko

Consider Equations (5) and (6) of Timoshenko model, do separation of variables to obtain eigenvalue problem

\[-u'' + \psi' = \lambda u, \quad (13)\]
\[-\frac{1}{\beta}u'' - u' + \psi = \frac{\lambda}{\alpha} \psi, \quad (14)\]

with the boundary conditions given by

\[u(0) = \psi(0) = u'(1) - \psi(1) = \psi'(1) = 0. \quad (15)\]

To calculate eigenvalues and eigenfunctions use method in [VV06].
To calculate eigenvalues for **Shear-T model**, use eigenvalue problem for Timoshenko with $\lambda = 0$ in equation (14).

To justify this, replace $\frac{1}{\alpha}$ by $\frac{\gamma}{\beta}$ and let $\gamma = 0$. ($\lambda$ depends continuously on $\gamma$.)

Frequency equation:

$$
\left( \frac{\lambda + \mu^2}{\lambda - \omega^2} + \frac{\lambda - \omega^2}{\lambda + \mu^2} \right) \cosh \mu \cos \omega + \left( \frac{\omega \mu - \mu}{\omega} \right) \sinh \mu \sin \omega = 2,
$$

but with

$$
\omega^2 = \frac{\lambda}{2} \left( \sqrt{1 + \frac{4\beta}{\lambda}} + 1 \right) \quad \text{and} \quad \mu^2 = \frac{\lambda}{2} \left( \sqrt{1 + \frac{4\beta}{\lambda}} - 1 \right).
$$
Comparison of Shear-T and Timoshenko eigenvalues

\[ \beta_{LA52} = 50. \]
For Timoshenko model \( \gamma = 0.25 \) and \( \gamma = 0 \) for Shear-T model.

<table>
<thead>
<tr>
<th>LA-52: North-South oscillation</th>
<th>Timoshenko model</th>
<th>Shear-T model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>( \lambda_k )</td>
<td>( \lambda_k )</td>
</tr>
<tr>
<td>1</td>
<td>0.2190</td>
<td>0.2232</td>
</tr>
<tr>
<td>2</td>
<td>5.3522</td>
<td>5.8336</td>
</tr>
<tr>
<td>3</td>
<td>27.3517</td>
<td>30.4359</td>
</tr>
<tr>
<td>4</td>
<td>69.5214</td>
<td>78.4895</td>
</tr>
<tr>
<td>5</td>
<td>132.8139</td>
<td>150.5247</td>
</tr>
<tr>
<td>6</td>
<td>201.4049</td>
<td>244.7589</td>
</tr>
</tbody>
</table>
Beam models for high-rise structures

**Adapted Timoshenko model**

\[
\begin{align*}
\rho^* \partial_t^2 u &= \partial_x S + P, \quad (16) \\
\rho^* \partial_t^2 w &= \partial_x V + Q, \quad (17) \\
\frac{\rho^*}{\alpha} \partial_t^2 \phi &= V + \partial_x M + S \partial_x w, \quad (18) \\
M &= \frac{1}{\beta} \partial_x \phi, \quad (19) \\
V &= \partial_x w - \phi, \quad (20) \\
S &= \frac{1}{\gamma} \partial_x u. \quad (21)
\end{align*}
\]
Parameter $\rho^*$

- Entire structure cannot be considered as a beam.
- Seems reasonable that part of building may be modelled as beam. (Reinforced concrete frames, steel frames and shear walls are mentioned in [RM05].)
- Additional mass that does not contribute to stiffness of the structure is present.
- Let $\rho_{RM}$ denote mass per unit length used in [RM05], then $\rho_{RM} > \rho_A$, where $\rho_A$ is mass per unit length of the “beam”.
- Let $\rho^* = \frac{\rho_{RM}}{\rho A}$, then $\rho^* > 1$. 
Only consider transverse vibration.

$$S = \mu (1 - x), \quad \mu = \frac{\rho g \ell}{G K^2} \ll 0.1.$$ 

A force density considered in Wang, Fung and Huang, 2001 but not in [RM05].

Effect of $S$ is hardly noticable.
Adapted Timoshenko model

\[ \rho^* \partial_t^2 w = \partial_x V, \]  \hspace{1cm} (22)

\[ \frac{\gamma \rho^*}{\beta} \partial_t^2 \phi = V + \partial_x M + S \partial_x w. \]  \hspace{1cm} (23)

Note that \( \frac{1}{\alpha} \) was replaced by \( \frac{\gamma}{\beta} \).

\( w(0, t) = w_E(t), \ u(0, t) = \phi(0, t) = 0. \)

\( M(1, t) = 0 \) and \( V(1, t) = 0. \)

Earthquake induced oscillations

- The force density \( Q = 0. \)
- In general \( u(0, t) \neq 0. \)
Equivalent problem

The earthquake model problem is equivalent to an artificial “wind problem” for a cantilever beam.

The boundary condition \( w(0, t) = w_E(t) \) can be homogenized: Let \( \tilde{w}(x, t) = w(x, t) - w_E(t)y(x) \) and \( \tilde{V} = \partial_x \tilde{w} - \phi \).

Equations (22) and (23) are transformed as follows

\[
\rho^* \partial_t^2 \tilde{w} = \partial_x \tilde{V} - \rho^* w_E - \rho^* \ddot{w}_E y, \tag{24}
\]

\[
\frac{\gamma \rho^*}{\beta} \partial_t^2 \phi = \tilde{V} + w_E y' + \partial_x M - \partial_x w_S, \tag{25}
\]

where \( y(x) = 1 + x - \frac{1}{2} x^2 \).
Boundary conditions:

\[ y(0) = 1 \text{ implies} \]

\[ \dddot{w}(0, t) = w_E(t) - w_E(t)y(0) = 0. \]

At the top

\[ \dddot{V}(1, t) = V(1, t) - w_E(t)y'(1) = V(1, t) = 0. \]

The other boundary conditions remain unchanged, i.e.

\[ M(1, t) = 0 \text{ and } \phi(0, t) = 0. \]

We now have a model problem for a cantilever beam.
Shear-M model

It is derived from a model in Miranda, 1999 for a building in equilibrium subjected to a distributed load $Q$ (equivalent problem). A shear beam is combined with an Euler-Bernoulli (flexural) beam.

$$\rho^* \partial_t^2 w - \sigma \partial_x^2 w + \frac{1}{\beta} \partial_x^4 w = Q,$$

where $\sigma = \frac{G_s A_s}{G A \kappa^2}$. (26)

In [RM05] the boundary conditions are not discussed. At $x = 0$ may use the boundary conditions for Rayleigh and at the top

$$\partial_x^2 w(1, t) = 0$$

and

$$\partial_x w(1, t) - \frac{1}{\beta \sigma} \partial_x^3 w(1, t) = 0.$$

Note that gravity is neglected in this model.
Stiffness ratio parameter in [RM05]: $\alpha_M = \beta \sigma$.

**Eigenvalue problem**

\[
u^{(4)} - \alpha_M u'' - \lambda \alpha_M u = 0, \quad \text{with}\]
\[
u(0) = u'(0) = 0,
\[
\frac{1}{\alpha_M} u'''(1) - u'(1) = 0,
\]
\[
u''(1) = 0.
\]

- Authors make use of their model to obtain the values of the parameters.
- Values of $\beta$ and $\sigma$ are not given separately in article - only $\alpha_M$ is given.
From the boundary conditions we also obtain the following frequency equation

\[
\left(2\frac{\mu^2\omega^2}{\beta} - \omega^2 + \mu^2\right) \cosh \mu \cos \omega \\
+ \left(2\mu\omega - \frac{\mu^3\omega}{\beta} + \frac{\mu\omega^3}{\beta}\right) \sinh \mu \sin \omega \\
+ \frac{\mu^4 + \omega^4}{\beta} - \mu^2 + \omega^2 = 0, \quad \text{with}
\]

\[
\mu^2 = \frac{\beta}{2} \left(1 + \sqrt{1 + \frac{4\lambda}{\beta}}\right) \quad \text{and} \quad \omega^2 = \frac{\beta}{2} \left(-1 + \sqrt{1 + \frac{4\lambda}{\beta}}\right).
\]
Comparison of two buildings using data from [RM05].

<table>
<thead>
<tr>
<th></th>
<th>LA-52</th>
<th>LA-54</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
<td>±200 m</td>
<td>±200 m</td>
</tr>
<tr>
<td>Floor dimensions</td>
<td>48 m × 48 m</td>
<td>60 m × 37 m</td>
</tr>
<tr>
<td>$\alpha_M$</td>
<td>$\alpha_{M,NS} = 7.8^2$</td>
<td>$\alpha_{M,NS} = 27.5^2$</td>
</tr>
<tr>
<td></td>
<td>$\alpha_{M,EW} = 6.6^2$</td>
<td>$\alpha_{M,EW} = 30^2$</td>
</tr>
<tr>
<td>Fundamental period</td>
<td>$T_{NS} = 5.8$</td>
<td>$T_{NS} = 6.2$</td>
</tr>
<tr>
<td></td>
<td>$T_{EW} = 6$</td>
<td>$T_{EW} = 5.2$</td>
</tr>
<tr>
<td>Peak ground acceleration</td>
<td>$PGA_{NS} = 165$</td>
<td>$PGA_{NS} = 165$</td>
</tr>
<tr>
<td></td>
<td>$PGA_{EW} = 109$</td>
<td>$PGA_{EW} = 98$</td>
</tr>
<tr>
<td>Peak roof acceleration</td>
<td>$PRA_{NS} = 389$</td>
<td>$PRA_{NS} = 177$</td>
</tr>
<tr>
<td></td>
<td>$PRA_{EW} = 220$</td>
<td>$PRA_{EW} = 139$</td>
</tr>
</tbody>
</table>
Simulation

- Nature of the disturbance should be taken into account - will determine number of modes involved. (If manner of excitation is such that only first mode is considered, then Euler-Bernoulli beam may still be fine.)

- Earthquake models: don’t know how many modes are involved - simulation is necessary.

- To investigate effect of disturbance our preliminary experiment was to simulate each model separately to observe the transient response of the structure.
Transient response of a building due to earthquake using Timoshenko model. Full period of the ground disturbance \( \tau_g = 8 \), \( w(0, t) = w_E = D \sin(Ct) \).
Illustration of effect of $\beta$ using Timoshenko model

$\beta = 50$ (in red) v.s. $\beta = 800$ (in blue).
Comparison of models

Consider the motion of top of building for full period of ground motion.

\[ \beta = 50 \quad \text{and} \quad \beta = 800 \]

Timoshenko (blue) v.s. Shear-T (red)
Results

\[ \beta = 50 \]

\[ \beta = 800 \]

Timoshenko (blue) v.s. Rayleigh (red)
Conclusion

- Rayleigh and Euler-Bernoulli only for $300 < \beta < 1200$.
- Shear-T compares well to Timoshenko - but difficulty in programming and no gain.
- Shear-M cannot be compared to Timoshenko using [RM05] data. Solution: Artificial building or data from another article.
END

Thank you