On the zeros of Meixner and Meixner-Pollaczek polynomials

Alta Jooste

University of Pretoria

SANUM 2016, University of Stellenbosch

March 22, 2016
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   - Quasi-orthogonal Meixner polynomials

4 Meixner-Pollaczek polynomials
   - Quasi-orthogonal Meixner-Pollaczek polynomials
Orthogonal polynomials

To define families of orthogonal polynomials, we use a scalar product

\[ \langle f, g \rangle := \int_a^b f(x)g(x) \, d\phi(x), \]

positive measure \( d\phi(x) \) supported on \([a, b], \ a, b \in \mathbb{R}.\)

A sequence of real polynomials \( \{p_n\}_{n=0}^N, \ N \in \mathbb{N} \cup \{\infty\}, \) is orthogonal on \((a, b)\) with respect to \(d\phi(x)\) if

\[ \langle p_n, p_m \rangle = 0 \text{ for } m = 0, 1, \ldots, n - 1. \]
Orthogonal polynomials

If $d\phi(x)$ is absolutely continuous and $d\phi(x) = w(x)dx$,

$$\int_{a}^{b} p_n(x)p_m(x)w(x)dx = 0 \text{ for } m = 0, 1, \ldots, n - 1$$

$\{p_n\}$ is orthogonal on $(a, b)$ w.r.t. the weight $w(x) > 0$.

If the weight is discrete and $w_j = w(j)$, $j \in \mathbb{L} \subset \mathbb{Z}$,

$$\sum_{j \in \mathbb{L}} p_n(j)p_m(j) w_j = 0 \text{ for } m = 0, 1, \ldots, n - 1$$

and the sequence $\{p_n\}$ is **discrete orthogonal**.

In the classical case: $\mathbb{L} = \{0, 1, \ldots, N\}$. 
Properties of orthogonal polynomials

(i) Three-term recurrence relation

\[(x - B_n)p_{n-1}(x) = A_np_n(x) + C_np_{n-2}(x), \ n \geq 1\]

\[p_{-1}(x) = 0; \ A_n, B_n, C_n \in \mathbb{R}; \ A_{n-1}C_n > 0, \ n = 1, 2, \ldots ;\]

(ii) \(p_n\) has \(n\) real, distinct zeros in \((a, b)\);

(iii) Classic interlacing of zeros

The zeros of \(p_n\) and \(p_{n-1}\) separate each other:

\[a < x_{n,1} < x_{n-1,1} < x_{n,2} < \cdots < x_{n-1,n-1} < x_{n,n} < b.\]
Introduction

Background

Meixner polynomials

Meixner-Polaczek polynomials

Orthogonality and quasi-orthogonality

Polynomials are orthogonal for specific values of their parameters, e.g.

Jacobi polynomials \( P_n^{\alpha,\beta} \):

orthogonal on \([-1, 1]\) w.r.t \(w(x) = (1 - x)^\alpha (1 + x)^\beta\) for \(\alpha, \beta > -1\).

Deviation from restricted values of the parameters results in zeros departing from interval of orthogonality

Question: Do polynomials with "shifted" parameters retain some form of orthogonality that explains the amount of zeros that remain in the interval of orthogonality?
Orthogonality and quasi-orthogonality

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- Deviation from restricted values of the parameters results in zeros departing from interval of orthogonality.

- Question: Do polynomials with "shifted" parameters retain some form of orthogonality that explains the amount of zeros that remain in the interval of orthogonality?
Quasi-orthogonality (Riesz, 1923)

A sequence of polynomials \( \{ R_n \}_{n=0}^N \) is **quasi-orthogonal** of order \( k \) with respect to \( w(x) \) on \([a, b]\) if

\[
\int_a^b x^m R_n(x) w(x) \, dx \begin{cases} 
= 0 & \text{for } m = 0, 1, \ldots, n - k - 1 \\
\neq 0 & \text{for } m = n - k.
\end{cases}
\]

Note that \( n = k + 1, k + 2, \ldots \).
Lemma 1
Let \( \{p_n\} \) be orthogonal on \([a, b]\) with respect to \(w(x)\). A necessary and sufficient condition for a polynomial \(R_n\) to be quasi-orthogonal of order \(k\) on \([a, b]\) with respect to \(w(x)\), is that

\[
R_n(x) = c_0 p_n(x) + c_1 p_{n-1}(x) + \cdots + c_k p_{n-k}(x)
\]

where the \(c_i\)'s are numbers which can depend on \(n\) and \(c_0 c_k \neq 0\).

Lemma 2
If \(\{R_n\}\) are real polynomials that are quasi-orthogonal of order \(k\) with respect to \(w(x)\) on an interval \([a, b]\), then at least \((n - k)\) zeros of \(R_n(x)\) lie in the interval \([a, b]\).
Meixner polynomials (Josef Meixner, 1934)

\[ M_n(x; \beta, c) = (\beta)_n \sum_{k=0}^{n} \frac{(-n)_k (-x)_k (1 - \frac{1}{c})^k}{(\beta)_k k!} \]

\( \beta, c \in \mathbb{R}, \beta \neq -1, -2, \ldots, -n + 1, c \neq 0 \).

( )\(_k\) is the Pochhammer symbol

\[ (a)_k = a(a+1)\ldots(a+k-1), \quad k \geq 1 \]

\[ (a)_0 = 1 \text{ when } a \neq 0 \]
Meixner polynomials

For $0 < c < 1$, $\beta > 0$,

$$\sum_{j=0}^{\infty} \frac{c^j(\beta)_j}{j!} M_m(j; \beta, c) M_n(j; \beta, c) = 0, \quad m = 1, 2, \ldots, n - 1,$$

hence the zeros are real, distinct and in $(0, \infty)$.

- $\frac{c^j(\beta)_j}{j!}$ constant on $(j, j + 1), j = 0, 1, 2, \ldots$;
- zeros are separated by mass points $j = 0, 1, 2, \ldots$. 
Meixner polynomials satisfy the difference equation:

\[ c(x+\beta)M_n(x+1; \beta, c) = \left(n(c-1)+x+(x+\beta)c\right)M_n(x; \beta, c) - xM_n(x-1; \beta, c). \]

**Krasikov, Zarkh (2009):** Suppose \( p_n(x) \) satisfies

\[ p_n(x + 1) = 2A(x)p_n(x) - B(x)p_n(x - 1) \]

and \( B(x) > 0 \) for \( x \in (a, b) \), then \( M(p_n) > 1 \).

\( M(p_n) \equiv \text{minimum distance between the zeros of } p_n(x). \)

- True for Hahn, Meixner, Krawtchouk and Charlier polynomials;
- Hahn polynomials: Levit (1967);
**Difference equation**

Meixner polynomials satisfy the difference equation:

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- True for Hahn, Meixner, Krawtchouk and Charlier polynomials;
- Hahn polynomials: Levit (1967);
As a consequence:
Zeros of $p_n(x - 1)$, $p_n(x)$ and $p_n(x + 1)$ interlace.

Zeros of $M_4(x - 1, 5; 0.45)$, $M_4(x, 5; 0.45)$ and $M_4(x + 1, 5; 0.45)$. 
Let \( 0 < \beta < 1, 0 < c < 1 \). By iterating the recurrence relation

\[
M_n(x; \beta - 1, c) = M_n(x; \beta, c) - nM_{n-1}(x; \beta, c),
\]

we obtain

\[
M_n(x; \beta - k, c) = c_0 M_n(x; \beta, c) + c_1 M_{n-1}(x; \beta, c) + \cdots + c_k M_{n-k}(x; \beta, c)
\]

and

- \( M_n(x; \beta - k, c) \) is quasi-orthogonal of order \( k \) for \( k \in \{1, 2, \ldots, n - 1\} \);
- at least \( n - k \) zeros remain in \((0, \infty)\).

To obtain relations necessary to prove our results, we use a Maple program by Vidunas.
Quasi-orthogonality of order 1

Theorem: If \(0 < c < 1\) and \(0 < \beta < 1\), then the smallest zero of \(M_n(x; \beta - 1, c)\) is negative.

Zeros of \(M_3(x, 0.4; 0.6)\) and \(M_3(x, 0.4 - 1; 0.6)\).

Interlacing results between the zeros of Quasi-orthogonal Meixner and Meixner polynomials were studied in 2015 [Driver, AJ, submitted 2015].
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Quasi-orthogonality of order 2

Theorem: If $0 < c < 1$, $0 < \beta < 1$ and $n > \frac{\beta - 2}{c - 1}$ then all the zeros of $M_n(x; \beta - 2, c)$ are nonnegative and simple.

For $\beta = 0.5$ and $c = 0.6$, $\frac{\beta - 2}{c - 1} = 3.75$.

Zeros of $M_4(x, 0.5 - 2; 0.6)$ and $M_3(x, 0.5 - 2; 0.6)$. 
Zeros of $M_4(x, 0.5 - 2; 0.6)$ and $M_3(x, 0.5 - 2; 0.6)$.
Definition of the (monic) Meixner polynomials,

\[
M_n(x; \beta, c) = (\beta)^n \left(\frac{c}{c-1}\right)^n \sum_{k=0}^{n} \frac{(-n)_k(-x)_k(1 - \frac{1}{c})^k}{(\beta)_k k!}
\]

Let \( c = e^{2i\phi}, x = -\lambda - ix \) and \( \beta = 2\lambda \), to obtain the Meixner-Pollaczek polynomials

\[
P^\lambda_n(x; \phi) = i^n (2\lambda)^n \left(\frac{e^{2i\phi}}{e^{2i\phi} - 1}\right)^n \sum_{k=0}^{n} \frac{(-n)_k(\lambda + ix)_k(1 - \frac{1}{e^{2i\phi}})^k}{(2\lambda)_k k!}.
\]

For \( n \in \mathbb{N}, \lambda > 0, 0 < \phi < \pi \), \( P^\lambda_n(x; \phi) \) are orthogonal on \(( -\infty, \infty)\) w.r.t. \( e^{(2\phi-\pi)x} |\Gamma(\lambda + ix)|^2 \).
Meixner-Pollaczek polynomials

Definition of the (monic) Meixner polynomials,

\[ M_n(x; \beta, c) = (\beta)_n \left( \frac{c}{c - 1} \right)^n \sum_{k=0}^{n} \frac{(-n)_k (-x)_k (1 - \frac{1}{c})^k}{(\beta)_k k!} \]

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\[ P_\lambda^n(x; \phi) = i^n (2\lambda)_n \left( \frac{e^{2i\phi}}{e^{2i\phi} - 1} \right)^n \sum_{k=0}^{n} \frac{(-n)_k (\lambda + ix)_k (1 - \frac{1}{e^{2i\phi}})^k}{(2\lambda)_k k!}. \]

For \( n \in \mathbb{N}, \lambda > 0, 0 < \phi < \pi \),

\( P_\lambda^n(x; \phi) \) are orthogonal on \(( -\infty, \infty)\) w.r.t. \( e^{(2\phi-\pi)x} |\Gamma(\lambda + ix)|^2 \).
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Let \( c = e^{2i\phi}, x = -\lambda - ix \) and \( \beta = 2\lambda \), to obtain the Meixner-Pollaczek polynomials

\[ P_n^\lambda(x; \phi) = i^n (2\lambda)_n \left( \frac{e^{2i\phi}}{e^{2i\phi} - 1} \right)^n \sum_{k=0}^{n} \frac{(-n)_k (\lambda + ix)_k (1 - \frac{1}{e^{2i\phi}})^k}{(2\lambda)_k k!}. \]

For \( n \in \mathbb{N}, \lambda > 0, 0 < \phi < \pi \),
\( P_n^\lambda(x; \phi) \) are orthogonal on \((-\infty, \infty)\) w.r.t. \( e^{(2\phi-\pi)x} |\Gamma(\lambda + ix)|^2 \).
Theorem
For $0 < \lambda < 1$ and $k = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$ fixed, the polynomial $P_{n}^{\lambda-k}(x; \phi)$ is quasi-orthogonal of order $2k$ with respect to $e^{(2\phi-\pi)x}|\Gamma(\lambda + ix)|^2$ on $(-\infty, \infty)$ and therefore has at least $n - 2k$ real zeros.

Contiguous relations are used to find

$$P_{n}^{\lambda-1}(x; \phi) = P_{n}^{\lambda}(x; \phi) - n \cot \phi P_{n-1}^{\lambda}(x; \phi) + \frac{n(n-1)}{4 \sin^2 \phi} P_{n-2}^{\lambda}(x; \phi)$$

By iteration, $P_{n}^{\lambda-k}(x; \phi)$ can be written as a linear combination of $P_{n}^{\lambda}(x, \phi), P_{n-1}^{\lambda}(x, \phi), \ldots, P_{n-2k}^{\lambda}(x, \phi)$ and we can apply Lemmas 1 and 2.
Zeros of $P_{4}^{0.4}(x; 1.6)$ and $P_{4}^{0.4-1}(x; 1.6)$. 
Thank you